AD-A102 581 CHICAGO UNIV IL DEPT OF GEOPHYSICAL SCIENCES F/G 20/11
INVERSE PROBLEM FOR THE VIBRATING BEAM IN THE FREE/CLAMPED CONF--ETC(U)
1979 V BARCILON N00014-76-C-0034

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Inverse Problem for the Vibrating

Beam in the Free/clamped Configuration

Contract N00014-76-C-0034

by

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ABSTRACT

We consider the problem of reconstructing the flexural rigidity $r(x)$ and the density $\rho(x)$ of a beam. The unknown beam is assumed to have a free left end and a clamped right one. The data consist of the displacement and angle of the center line of the free left end following an initial impulse. The information content of this seismogram-like impulse response is equivalent to three spectra $\{\omega_n\}$, $\{\nu_n\}$, $\{\mu_n\}$ and two gross constants F_1, F_2. This data do not specify the structure of the vibrating beam uniquely, but rather a class of beams. All the beams in this class share the same structure over that portion of their length which is actually set in motion; they can differ over the portion which is stationary. A method for constructing $r(x)$ and $\rho(x)$ is presented. It consists of two steps: first $\rho(x)$ and $r(x)$ are determined over a small interval $(0,x)$ adjacent to the free left end. Next, this known portion of the beam is stripped-off and the response of the resulting truncated beam is computed via the initial data. The procedure is then repeated. Finally, the question of the existence of a solution is discussed. More specifically, conditions on $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ are given which insure that $r(x)$ and $\rho(x)$ are physically meaningful.

LIST OF SYMBOLS

a, b, c, d, e	coefficients in boundary conditions of canonical 4th order problem
f :	forcing; flaccidity of clothespin (discrete beam)
h :	arbitrary function in eigenfunction expansion
i :	index
j :	index
k, k^*:	constants of proportionality between eigenfunctions
l :	separation of clothespins (discrete beam)
m :	mass of clothespin (discrete beam); index
n :	index
p, q	variable in canonical 4th order problem
r :	flexural rigidity
s :	Laplace transform variable
t :	time; dummy variable of integration
u, v	fundamental solutions
x :	coordinate
y :	generic displacement of beam; modal shape
A, B	coefficients in canonical 4th order problem
D :	auxiliary function associated with free/clamped configuration
F_o, F_1, F_2:	zeroth, first and second moment of flaccidity
G :	Green's function
I :	auxiliary function associated with clamped/clamped configuration
J :	auxiliary function associated with non-self adjoint/clamped configuration
K :	auxiliary function associated with Rayleigh/clamped configuration
L :	length of beam
N :	Green's function for non-self adjoint eigenvalue problem
P, Q	logarithmic derivatives of p and q
U_n:	eigenfunctions of canonical 4th order problem
W :	Wronskian
X :	$\frac{s^{\frac{1}{4}}}{\sqrt{2}} \int_0^x \left(\frac{\rho}{r} \right)^{\frac{1}{4}} dx$, WKBJ variable
Y :	auxiliary variable associated with supported/clamped configuration
Z :	dimensionless variable for homogeneous beam.

$\left.\begin{matrix}\alpha\\ \beta\end{matrix}\right\}$ generic parameters

η : forcing; eigenfunctions in (3.7)

θ : slope of center line

λ_n : eigenfrequencies of clamped/clamped beam

μ_n : eigenfrequencies of supported/clamped beam

ν_n : eigenfrequencies of non-self adjoint/clamped beam

ξ : *coordinate in canonical* 4th order problem

$$\xi = \int_0^x \left(\frac{\rho}{r} \right)^{\frac{1}{4}} dx$$

ϖ : X(L)

ρ : density

σ_n : eigenfrequencies of Rayleigh/clamped beam

τ : stress

ϕ : fundamental solution

χ : moment

ψ : fundamental solution

$\omega;\omega_n$: frequency; eigenfrequencies of free/clamped beam

Ξ : $\displaystyle\int_0^L \left(\frac{\rho}{r} \right)^{\frac{1}{4}} dx$

Θ : auxiliary function for non self-adjoint/clamped configuration

Λ_n : eigenvalues in (3.5)

$\left.\begin{matrix}\Phi\\ \Psi\end{matrix}\right\}$ fundamental solutions

Ω : Green's function for free/clamped beam.

INTRODUCTION

This paper is devoted to a review of the current status of the inverse problem for a vibrating beam. This inverse problem consists in reconstructing the flexural rigidity $r(x)$ and the density $\rho(x)$ as a function of position x from data associated with natural frequencies of vibration of the beam.

The reconstruction of $r(x)$ and $\rho(x)$ is made without assuming any *a-priori* knowledge about the structure of the beam. Consequently, this approach differs from the traditional approach for the geophysical problem dealing with the reconstruction of the internal structure of the Earth from data on toroidal and spheroidal oscillations. There, the spectral data are used to *correct* a model which incorporates knowledge from travel-time, surface waves, etc... Thus, the Backus-Gilbert technique, which is ideally suited for such correcting tasks, is not suitable here.

The above mentioned geophysical problem is a primary motivation for the investigation of the inverse problem for a vibrating beam. Indeed, if one neglects the gravitational force, the rotation, the oblateness, etc... and considered the Earth as an elastic, radially stratified sphere, then its normal modes are of the two types previously alluded, namely the spheroidal modes and the toroidal modes. The latter are governed by a 2nd order equation of Sturm-Liouville type whereas the former are governed by a 4th order system. The bulk and shear modulii as well as the density, which characterize the elastic properties of this idealized Earth are assumed unknown. The inverse problem consists in retrieving these characteristics. In order to understand this difficult inverse problem, it might be helpful to consider simpler inverse problems, and in particular, simpler inverse problems associated with 4th order equations since the inverse Sturm-Liouville

problem is well understood. The inverse problem for the vibrating beam is an ideal candidate.

This problem can also be viewed as an example of an inverse problem associated with a class of differential operators of order 2n, namely

$$L_{2n} = (-1)^n \alpha_o \frac{d}{dx} \alpha_1 \frac{d}{dx} \cdots \frac{d}{dx} \alpha_n \frac{d}{dx} \cdots \frac{d}{dx} \alpha_1 \frac{d}{dx} \alpha_o \quad ,$$

where all the α's are positive. For certain boundary conditions, the Green's functions associated with these operators are oscillating kernels. The theory of such kernels plays an important role in the solution of the inverse problem.

The outline of the paper is as follows: in §1, we shall introduce the terminology associated with the five eigenvalue problems which will be required in the sequel. Various properties of fundamental solutions will also be given. The question of the uniqueness of the inverse problem is taken up in §2. We shall show that the impulse response, which contains information equivalent to three spectra and two gross constants, determines a class of beams. The beams in this class have the same massive, flexible front end of length L, and a weightless, infinitely rigid back end of arbitrary length. By selecting that non-pathological beam without any back-end we shall achieve uniqueness. The actual construction of this beam is presented in §3. The construction procedure requires a discretization of the vibrating system: this is accomplished by replacing the beam by a series of massive clothespins of various stiffness connected to each others by weightless, infinitely rigid rods. Formulas for the characteristics of the clothespin nearest to the excitation are given. Once this clothespin

is determined, it is "stripped-off" the structure; the spectra for the reduced beam are found in terms of the given data. The same procedure is repeated until all of the characteristics are determined. This procedure fails if the given data is not *bona fide* data. The question of what constitutes *bona fide* data is also discussed.

PRELIMINARY RESULTS

We shall be concerned with various solutions of the equation

$$\frac{d^2}{dx^2}\, r(x)\, \frac{d^2y}{dx^2} = \omega^2\, \rho(x)\, y\ , \quad 0 < x < L\ . \tag{1.1}$$

The flexural rigidity $r(x)$ and the density $\rho(x)$ are non-negative functions, i.e.,

$$r(x) \geq 0\ ,$$
$$\rho(x) \geq 0\ .$$

In the section dealing with the uniqueness of the inverse eigenvalue problem, we shall assume that $r(x)$ and $\rho(x)$ are differentiable and in fact that they possess four derivatives. These requirements are very restrictive and can be relaxed in places.

We also find it convenient to write (1.1) as a system of first order differential equations, namely

$$\begin{aligned} \frac{dy}{dx} &= \theta \\ \frac{d\theta}{dx} &= \frac{1}{r}\,\tau \\ \frac{d\tau}{dx} &= -\,\chi \\ \frac{d\chi}{dx} &= -\,\omega^2\,\rho\, y \end{aligned} \tag{1.2}$$

The variables θ , τ and χ have a simple physical meaning: θ is the slope of the central line, τ is the stress and χ the moment.

Several fundamental solutions of (1.1) will be useful in the sequel. In particular, we shall define $\phi(x, \omega^2)$ and $\psi(x, \omega^2)$ thus:

$$\phi(0, \omega^2) - 1 = \phi'(0, \omega^2) = \phi''(0, \omega^2) = \phi'''(0, \omega^2) = 0 \tag{1.3a}$$

$$\psi(0, \omega^2) = \psi'(0, \omega^2) - 1 = \psi''(0, \omega^2) = \psi'''(0, \omega^2) = 0 \tag{1.3b}$$

when primes denote derivatives with respect to x . $\phi(x, \omega^2)$ and $\psi(x, \omega^2)$ are entire functions of ω^2 of order $\frac{1}{4}$.* To prove this assertion one would show (i) that the Taylor series expansions of ϕ and ψ are convergent throughout the entire complex ω^2-plane and (ii) that ϕ and ψ are dominated by entire functions of ω^2 of order $\frac{1}{4}$. The reader is referred to Titchmarsh (1962) p. 6-11 where a similar proof is given for the Sturm-Lionville case.

The growth of ϕ and ψ can also be seen from their asymptotic behaviors obtained via the WKBJ method, viz.

$$\phi(x, -s) \sim \left[\frac{\rho^3(0)\ r(0)}{\rho^3(x)\ r(x)}\right]^{\frac{1}{8}} \cos X \ \cosh X \tag{1.4a}$$

$$|s| \to \infty$$

$$\psi(x, -s) \sim \frac{1}{\sqrt{2}\ s^{\frac{1}{4}}} \left[\frac{\rho(0)\ r^3(0)}{\rho^3(x)\ r(x)}\right]^{\frac{1}{8}} \{\cos X \sinh X + \sin X \ \cosh X\} \tag{1.4b}$$

where

$$X = \frac{s^{\frac{1}{4}}}{\sqrt{2}} \int_0^x \left(\frac{\rho}{r}\right)^{\frac{1}{4}} dx\ . \tag{1.5}$$

*see Boas (1954) p. 8 for the definition of "order".

By replacing s by $-\omega^2$, we see that the entire functions ϕ and ψ are indeed of order $\frac{1}{4}$.

ϕ and ψ are particularly well suited for studying the vibrations of a beam with a free left end, i.e. a beam satisfying the boundary conditions

$$y'' = (ry'')' = 0 \quad \text{at} \quad x = 0. \tag{1.6}$$

Insofar as the right end is concerned, we shall exclusively deal with the clamped case, i.e.

$$y = y' = 0 \quad \text{at} \quad x = L. \tag{1.7a}$$

To that effect, we introduce two additional fundamental solutions of (1.1), say, u and v such that

$$u(L, \omega^2) = u'(L, \omega^2) = u''(L, \omega^2) - 1 = u'''(L, \omega^2) = 0, \tag{1.7b}$$

$$v(L, \omega^2) = v'(L, \omega^2) = v''(L, \omega^2) = v'''(L, \omega^2) - 1 = 0. \tag{1.7c}$$

Needless to say, u and v are also entire functions of ω^2 of order $\frac{1}{4}$.

The Wronskian $W(x, \omega^2)$ which is:

$$W(x, \omega^2) = \begin{vmatrix} \phi & \psi & u & v \\ \phi' & \psi' & u' & v' \\ \phi'' & \psi'' & u'' & v'' \\ \phi''' & \psi''' & u''' & v''' \end{vmatrix} \tag{1.8}$$

satisfies the following differential equation:

$$\frac{d}{dx}\,[r^2(x)\ W(x,\ \omega^2)] = 0\ . \qquad (1.9)$$

Consequently

$$r^2(x)\ W(x,\ \omega^2) = r^2(0)\ W(0,\ \omega^2)\ , \qquad (1.10a)$$

$$= r^2(L)\ W(L,\ \omega^2)\ . \qquad (1.10b)$$

But

$$W(0,\ \omega^2) = u''(0,\ \omega^2)\ v'''(0,\ \omega^2) - u'''(0,\ \omega^2)\ v''(0,\ \omega^2) \qquad (1.10c)$$

and

$$W(L,\ \omega^2) = \phi(L,\ \omega^2)\ \psi'(L,\ \omega^2) - \phi'(L,\ \omega^2)\ \psi(L,\ \omega^2)\ . \qquad (1.10d)$$

Therefore the Wronskian $W(x,\ \omega^2)$ vanishes whenever ω coincides with the eigenfrequency ω_n of the beam in the free/clamped configuration.

The eigenvalue problem

$$\left.\begin{aligned} &(ry_n'')'' = \omega_n^2\ \rho\ y_n\ , \\ &y_n'' = (ry_n'')' = 0\ , \quad \text{for} \quad x = 0\ , \\ &y_n = y_n' = 0\ , \quad \text{for} \quad x = L\ , \end{aligned}\right\} \qquad (1.11)$$

will be the central eigenvalue problem of the paper. However, other eigenvalue problems will occasionally be needed. They differ from (1.11) only insofar as the left boundary conditions are concerned: the right end is always as in (1.11), i.e. clamped. For instance, if the left end were

clamped, then we would have:

$$\left.\begin{aligned} &(ry_n'')'' = \lambda_n^2 \rho\, y_n \\ &y_n = y_n' = 0\ , \quad \text{for} \quad x = 0\ , \\ &y_n = y_n' = 0\ , \quad \text{for} \quad x = L\ . \end{aligned}\right\} \tag{1.12}$$

Of course the eigenfunctions of (1.12) are different from those of (1.11): nevertheless, we use the same notation, viz. y_n to denote both eigenfunctions since no confusion will arise. In fact, we shall only make use of the eigenfrequencies $\{\lambda_n\}$ and not of the eigenfunctions.

The other eigenvalue problems needed in the sequel are those associated with the supported/clamped configuration:

$$\left.\begin{aligned} &(ry_n'')'' = \mu_n^2 \rho\, y_n \\ &y_n = y_n'' = 0\ , \quad \text{at} \quad x = 0\ , \\ &y_n = y_n' = 0\ , \quad \text{at} \quad x = L\ , \end{aligned}\right\} \tag{1.13}$$

and the Rayleigh*/clamped configuration:

$$\left.\begin{aligned} &(ry_n'')'' = \sigma_n^2 \rho\, y_n \\ &y_n' = (ry_n'')' = 0\ , \quad \text{at} \quad x = 0\ , \\ &y_n = y_n' = 0\ , \qquad \text{at} \quad x = L\ . \end{aligned}\right\} \tag{1.14}$$

*In honor of Lord Rayleigh who touched upon it in his theory of sound (1945) p. 259.

The eigenvalue problems (1.11), (1.12), (1.13) and (1.14) are self-adjoint and physically meaningful. The next pair of eigenvalue problems are neither. However, they do enter in our analysis:

$$\left.\begin{aligned} &(ry_n'')'' = \nu_n^2\,\rho\,y_n\,, \\ &y_n = (ry_n'')' = 0 \quad \text{at} \quad x = 0\,, \\ &y_n = y_n' \qquad\;\; = 0 \quad \text{at} \quad x = L\,; \end{aligned}\right\} \tag{1.15a}$$

$$\left.\begin{aligned} &(ry_n'')'' = \nu_n^2\,\rho\,y_n\,, \\ &y_n' = y_n'' = 0 \quad \text{at} \quad x = 0\,, \\ &y_n = y_n' \;\; = 0 \quad \text{at} \quad x = L\,. \end{aligned}\right\} \tag{1.15b}$$

(1:15b) is the adjoint of (1.15a): hence they have the same eigenfrequencies.

More general boundary conditions could have been considered, but for the sake of presentation, I have restricted myself to the above simple configurations. The Table summarizes the notations and is convenient for future reference

Left Boundary Conditional	Eigenfrequencies	Configuration
$y = y' = 0$	λ_n	clamped
$y = y'' = 0$	μ_n	supported
$y = (ry'')' = 0$	ν_n	non self-adjoint
$y' = y'' = 0$	ν_n	non self-adjoint
$y' = (ry'')' = 0$	σ_n	Rayleigh
$y'' = (ry'')' = 0$	ω_n	free

Let us return to our fundamental solutions ϕ, φ, u and v and record their values for $\omega^2 = 0$:

$$\phi(x,\ 0) = 1\ , \tag{1.16a}$$

$$\phi(x,\ 0) = x\ , \tag{1.16b}$$

$$u(x,\ 0) = r'(L) \int_L^x \frac{(x-x')(x'-L)}{r(x')}\, dx'$$

$$+\ r(L) \int_L^x \frac{x-x'}{r(x')}\, dx'\ , \tag{1.16c}$$

$$v(x,\ 0) = r(L) \int_L^x \frac{(x-x')(x'-L)}{r(x')}\, dx'\ . \tag{1.16d}$$

As a result, we deduce from (1.10d) that

$$W(L,\ 0) = 1 \tag{1.17}$$

Making use of Hadamard's Factorization theorem for entire functions (Boas 1954, p. 22) for $W(L,\ \omega^2)$ which is also an entire of order $\frac{1}{4}$, we can write

$$W(L,\ \omega^2) = W(L,\ 0) \prod_{m=1}^{\infty} \left(1 - \frac{\omega^2}{\omega_m^2}\right)\ . \tag{1.18a}$$

or, on account of (1.17),

$$W(L,\ \omega^2) = \prod_{m=1}^{\infty} \left(1 - \frac{\omega^2}{\omega_m^2}\right)\ . \tag{1.18b}$$

The asymptotic behavior of the eigenfrequencies $\{\omega_n\}$ for n large, which we shall recall presently, guarantees the convergence of the infinite product. Inserting (1.18a) in (1.10b), we get

$$W(x, \omega^2) = \left[\frac{r(L)}{r(x)}\right]^2 \prod_m^{\infty} \left(1 - \frac{\omega^2}{\omega_m^2}\right) . \tag{1.19}$$

The fundamental solutions u and v will not be used in their present form. Rather they will enable us to define two new fundamental solutions, viz

$$\Phi(x, \omega^2) = r(x) \begin{vmatrix} \psi & u & v \\ \psi' & u' & v' \\ \psi'' & u'' & v'' \end{vmatrix} \tag{1.20}$$

and

$$\Psi(x, \omega^2) = r(x) \begin{vmatrix} \phi & u & v \\ \phi' & u' & v' \\ \phi'' & u'' & v'' \end{vmatrix} \tag{1.21}$$

It is a simple matter to check that Φ and Ψ are indeed solutions of (1) as well as that

$$\Phi''(0, \omega^2) = \Phi(L, \omega^2) = \Phi'(L, \omega^2) = 0 , \tag{1.22}$$

$$(r\Psi'')'(0, \omega^2) = \Psi(L, \omega^2) = \Psi'(L, \omega^2) = 0 . \tag{1.23}$$

We give the asymptotic behavior of Φ and Ψ obtained once again by means of a WKBJ approach

$$\Phi(x, -s) \sim \frac{1}{\sqrt{2}\; s^{\frac{3}{4}}} \cdot \frac{r^{\frac{3}{2}}(L)}{\rho^{\frac{1}{2}}(L)} \left[\frac{r^3(0)}{r(x)} \frac{\rho(0)}{\rho^3(x)}\right]^{\frac{1}{8}} . \tag{1.24a}$$

$$\cdot \{[\cos \varpi \sinh \varpi + \sin \varpi \cosh \varpi] \sin(X - \varpi) \sinh(X - \varpi)$$
$$+ \cos \varpi \cosh \varpi [\sin(X - \varpi) \cosh(X - \varpi) - \cos(X - \varpi) \sinh(X - \varpi)]\}$$

$$\Psi(x, -s) \sim \frac{1}{2s^{\frac{1}{2}}} \cdot \frac{r^{\frac{3}{2}}(L)}{\rho^{\frac{1}{2}}(L)} \left[\frac{r(0)}{r(x)} \frac{\rho^3(0)}{\rho^3(x)}\right]^{\frac{1}{8}} .$$

(1.24b)

$$\cdot \{2 \cos \varpi \cosh \varpi \sin(X - \varpi) \sinh(X - \varpi)$$

$$+ [\cos \varpi \sinh \varpi - \sin \varpi \cosh \varpi][\sin(X - \varpi) \cosh(X - \varpi) - \cos(X - \varpi) \sinh(X - \varpi)]\}$$

where

$$\varpi = \frac{s^{\frac{1}{4}}}{\sqrt{2}} \int_0^L \left(\frac{\rho}{r}\right)^{\frac{1}{4}} dx ,$$

and X is given in (1.5). Finally, by substituting (1.4a) and (1.4b) in (1.10d) we can deduce that

$$r^2(0)\, W(0, -s) \sim \frac{1}{2}\left[\frac{\rho(0)}{\rho(L)} \cdot r(0)\, r^3(L)\right]^{\frac{1}{2}} \{\cos^2 \varpi + \cosh^2 \varpi\} \qquad (1.25)$$

In view of (1.22), we can use the function Φ to solve the eigenvalue problems (1.11), (1.13) and (1.15b). In particular, from (1.13) we deduce that

$$\Phi(0, \omega^2) = \Phi(0, 0) \prod_m^{\infty} \left(1 - \frac{\omega^2}{\mu_m^2}\right)$$

and, making use of the formulas (1.16) which express the behavior near $x = 0$, we see that

$$\Phi(0, \omega^2) = r^2(L)\, F_2 \prod_m^{\infty} \left(1 - \frac{\omega^2}{\mu_m^2}\right) \qquad (1.26)$$

where F_2 is the second moment of the "flaccidity", viz.

$$F_2 = \int_0^L \frac{x^2}{r(x)}\, dx \,. \tag{1.27}$$

Similarly, in view of (1.23) and (1.15a) we can write

$$\Psi(0, \omega^2) = \Psi(0, 0) \prod_m^{\infty} \left(1 - \frac{\omega^2}{\nu_m^2}\right)$$

and on account of (1.16)

$$\Psi(0, \omega^2) = r^2(L)\, F_1 \prod_m^{\infty} \left(1 - \frac{\omega^2}{\nu_m^2}\right) \tag{1.28}$$

where F_1 is the first moment of the flaccidity:

$$F_1 = \int_0^L \frac{x}{r(x)}\, dx \,. \tag{1.29}$$

Finally, by similar means we can show that

$$\Phi'(0, \omega^2) = -r^2(L)\, F_1 \prod_m^{\infty} \left(1 - \frac{\omega^2}{\nu_m^2}\right) . \tag{1.30a}$$

$$r(0)\, \Phi'''(0, \omega^2) = r^2(L) \prod_m^{\infty} \left(1 - \frac{\omega^2}{\omega_m^2}\right) . \tag{1.30b}$$

The functions Φ and Ψ are linearly independent as long as ω differs from an eigenvalue ω_n of the beam in the free/clamped configuration. In this case we have

$$\Phi(x, +\omega_n^2) = k_n\, \Psi(x, \omega_n^2) \,. \tag{1.31}$$

The constants of proportionality k_n, can be deduced by specializing (1.31) to $x = 0$ and using (1.26) and (1.28):

$$k_n = \frac{F_2}{F_1} \prod_m^{\infty} \frac{1 - \frac{\omega_n^2}{\mu_m^2}}{1 - \frac{\omega_n^2}{\nu_m^2}} \quad . \tag{1.32}$$

To simplify the notations, we shall define

$$\Psi_n(x) \equiv \Psi(x, \omega_n^2) \quad . \tag{1.33}$$

Thus $\{\Psi_n, \omega_n\}$ are the eigensolutions of (1.11). We can also represent the eigenfunctions of (1.11) in terms of ϕ and ψ. Thus we have another formula akin to (1.31):

$$\frac{\psi(L, \omega_n^2)}{\phi(L, \omega_n^2)} \phi(x, \omega_n^2) - \psi(x, \omega_n^2) = k_n^* \Psi_n(x) \; . \tag{1.34}$$

It should be noted that from the definitions (1.20) and (1.21) of Φ and Ψ we can deduce that

$$\Phi''(L, \omega^2) = r'(L)\, \psi(L, \omega^2)$$

$$\Psi''(L, \omega^2) = r'(L)\, \phi(L, \omega^2)$$

As a result

$$\frac{\psi(L, \omega_n^2)}{\phi(L, \omega_n^2)} = \frac{\Phi''(L, \omega_n^2)}{\Psi''(L, \omega_n^2)} = k_n \tag{1.35}$$

and so (1.34) can be written as follows:

$$k_n \, \phi(x, \omega_n^2) - \psi(x, \omega_n^2) = k_n^* \, \Psi_n(x) \; . \qquad (1.34')$$

In order to find the values of k_n^*, we set $x = 0$ and use (1.28):

$$k_n^* = \frac{k_n}{r^2(L) \; F_1 \prod_m^{\infty} \left(1 - \frac{\omega_n^2}{\nu_m^2}\right)} \; . \qquad (1.36)$$

We consider next the asymptotic form of the eigenfrequencies of the five problems. It is easy to see that

$$\Phi(0, \lambda_m^2) \; \Psi'(0, \lambda_m^2) - \Phi'(0, \lambda_m^2) \; \Psi(0, \lambda_m^2) = 0 \qquad (1.37a)$$

$$\Phi(0, \mu_m^2) = 0 \qquad (1.37b)$$

$$\Psi(0, \nu_m^2) = \Phi'(0, \nu_m^2) = 0 \qquad (1.37c)$$

$$\Psi(0, \sigma_m^2) = 0 \qquad (1.37d)$$

$$(r\Phi'')' \; (0, \omega_m^2) = \Psi''(0, \omega_m^2) = 0 \qquad (1.37e)$$

As a matter of fact, (1.37b) and (1,37c) are equivalent to (1.26), (1.28) and (1.30). By replacing Φ and Ψ in the above equations by their asymptotic forms as given in (1.24a) and (1.24b) we deduce that

$$\left.\begin{aligned} \lambda_n^2 &\sim (n + \tfrac{1}{2})^4 \, \pi^4/\Xi^4 \\ \mu_n^2 &\sim (n + \tfrac{1}{4})^4 \, \pi^4/\Xi^4 \\ \nu_n^2 &\sim n^4 \, \pi^4/\Xi^4 \\ \sigma_n^2 &\sim (n - \tfrac{1}{2})^4 \, \pi^4/\Xi^4 \\ \omega_n^2 &\sim (n - \tfrac{1}{4})^4 \, \pi^4/\Xi^4 \end{aligned}\right\} \qquad (1.38)$$

where

$$\Xi = \int_0^L \left(\frac{\rho}{r} \right)^{1/4} dx \tag{1.39}$$

Other important properties of the eigenvalues will be needed. For instance, we shall exploit the fact that the eigenvalues λ_n, μ_n, ν_n, σ_n and ω_n are all simple eigenvalues. This result is a consequence of the theory of integral equations with oscillating kernels [Gantmacher and Krein, 1950]. We can only give the briefest outline here. The first step is to convert the eigenvalue problems (1.11), (1.12), (1.13), (1.14) and (1.15) into integral equations of the form

$$y(x) = \omega^2 \int_0^L G(x, t)\, \rho(t)\, y(t)\, dt . \tag{1.40}$$

If the kernels $G(x, t)$, which are the Green's functions for the various problems, satisfy the following properties:

$$\text{(i)} \qquad G\begin{pmatrix} x_1 & \dots & x_n \\ t_1 & \dots & t_n \end{pmatrix} \equiv \begin{vmatrix} G(x_1, t_1) & \dots & G(x_1, t_n) \\ \vdots & & \vdots \\ G(x_n, t\ ) & \dots & G(x_n, t_n) \end{vmatrix} \geq 0 \tag{1.41}$$

for all partitions

$$0 < \begin{matrix} x_1 < x_2 < \dots < x_n \\ t_1 < t_2 < \dots < t_n \end{matrix} < L , \qquad n = 1, 2, \dots$$

and

$$\text{(ii)} \qquad G\begin{pmatrix} x_1 & \dots & x_n \\ x_1 & \dots & x_n \end{pmatrix} > 0 \tag{1.42}$$

for all partitions and all values of n, amd

(iii) $G(x, t) > 0$ for $0 < x, t < L$ (1.43)

then the kernel $G(x, t)$ is said to be oscillating. For such kernels, Gantmacher (1936) [see also Gantmacher and Krein, 1950] showed that the eigenvalues of (1.40) are simple.

Gantmacher and Krein (1950) have shown that the Green's functions for four out of the five eigenvalue problems of interest to us are indeed oscillating kernels. These are the four self-adjoint problems (1.11), (1.12), (1.13) and (1.14): thus λ_n, μ_n, σ_n and ω_n are simple eigenvalues.

The non-self adjoint problems (1.15a) has not been considered. However, by means of a theorem of Karlin (1971), we can state that the Green's function for that problem satisfies 1.41, i.e. is "totally positive" in Karlin's parlance. The fact that the kernel of the integral equation is a "totally positive" Green's function implies that (1.42) is also satisfied: the proof of this assertion can be found in Krein and Finkelstein (1939). Thus we only need to prove that (1.43) holds.

It is a simple matter to check that the Green's function under consideration denoted by $N(x, t)$ is given by

$$N(x, t) = \Omega(x, t) - \frac{1}{F_1} \int_x^L \frac{z - x}{r(z)} \, dz \int_t^L \frac{z(z - t)}{r(z)} \, dz . \qquad (1.44)$$

where

$$\Omega(x, t) = \begin{cases} \displaystyle\int_x^L \frac{(z - x)(z - t)}{r(z)} \, dz , & x \geq t , \\ \displaystyle\int_t^L \frac{(z - x)(z - t)}{r(z)} \, dz , & x \leq t . \end{cases} \qquad (1.45)$$

Consequently,

$$N(t, t) = \int_t^L \frac{(z - t)^2}{r(z)} dz - \frac{1}{F_1} \int_t^L \frac{z - t}{r(z)} dz \int_t^L \frac{z(z - t)}{r(z)} dz$$

or better still

$$N(t, t) = \left[1 - \frac{\int_t^L \frac{z\,dz}{r(z)}}{F_1} \right] \int_t^L \frac{(z - t)^2}{r(z)} dz$$

$$+ \frac{t}{F_1}\left[\int_t^L \frac{z^2 dz}{r(z)} \int_t^L \frac{dz}{r(z)} - \left(\int_t^L \frac{z\,dz}{r(z)} \right)^2 \right] \geq 0 \qquad (1.46)$$

Now, over the interval $(0, t)$

$$r(x) \frac{\partial^2 N}{\partial x^2} = - \frac{1}{F_1} \int_t^L \frac{(z - t)z}{r(z)} dz \leq 0 \qquad (1.47)$$

and hence N is concave downward, which together with (1.46) implies that

$$N(x, t) \geq 0 \quad \text{for} \quad 0 \leq x \leq t .$$

Over the remaining interval (t, L) ,

$$r(x) \frac{\partial^2 N}{\partial x^2} = x - t - \frac{1}{F_1} \int_t^L \frac{(z - t)z}{r(z)} dz$$

or, better still

$$r(x) \frac{\partial^2 N}{\partial x^2} = x - x_0(t) \qquad (1.48)$$

where

$$x_0(t) = \frac{1}{F_1}\left[t \int_0^t \frac{z\,dz}{r(z)} + \int_t^L \frac{z^2\,dz}{r(z)} \right] \tag{1.49}$$

Clearly

$$t \leq x_0(t) \leq L$$

and so $N(x, t)$ is respectively concave downward and upward over $(t, x_0(t))$ and $(x_0(t), L)$. Note that

$$N(x_0(t), t) = \Omega(x_0(t), t) - [x_0(t) - t] \int_{x_0(t)}^{L} \frac{z - x_0(t)}{r(z)}\,dz$$

i.e.

$$N(x_0(t), t) = \int_{x_0(t)}^{L} \frac{(z - x_0)^2}{r(z)}\,dz \geq 0$$

and so

$$N(x, t) \geq 0 \qquad \text{for} \qquad 0 \leq x \leq x_0(t) .$$

See Fig. 1.

We obtain the desired result, by noting that over the interval $(x_0(t), L)$, $N(x, t)$ must lie above its tangents. The x-axis being one such tangent, N must therefore be positive in this interval. Consequently

$$N(x, t) > 0 , \qquad \text{for} \qquad 0 < x, t < L ,$$

i.e $N(x, t)$ is an oscillating kernel and hence the eigenvalues ν_n are simple eigenvalues.

Consequently,

$$N(t, t) = \int_t^L \frac{(z - t)^2}{r(z)} dz - \frac{1}{F_1} \int_t^L \frac{z - t}{r(z)} dz \int_t^L \frac{z(z - t)}{r(z)} dz$$

or better still

$$N(t, t) = \left[1 - \frac{\int_t^L \frac{z\,dz}{r(z)}}{F_1} \right] \int_t^L \frac{(z - t)^2}{r(z)} dz$$

$$+ \frac{t}{F_1}\left[\int_t^L \frac{z^2 dz}{r(z)} \int_t^L \frac{dz}{r(z)} - \left(\int_t^L \frac{z\,dz}{r(z)} \right)^2 \right] \geq 0 \qquad (1.46)$$

Now, over the interval $(0, t)$

$$r(x) \frac{\partial^2 N}{\partial x^2} = - \frac{1}{F_1} \int_t^L \frac{(z - t)z}{r(z)} dz \leq 0 \qquad (1.47)$$

and hence N is concave downward, which together with (1.46) implies that

$$N(x, t) \geq 0 \quad \text{for} \quad 0 \leq x \leq t .$$

Over the remaining interval (t, L) ,

$$r(x) \frac{\partial^2 N}{\partial x^2} = x - t - \frac{1}{F_1} \int_t^L \frac{(z - t)z}{r(z)} dz$$

or, better still

$$r(x) \frac{\partial^2 N}{\partial x^2} = x - x_0(t) \qquad (1.48)$$

Insert Figure 1.

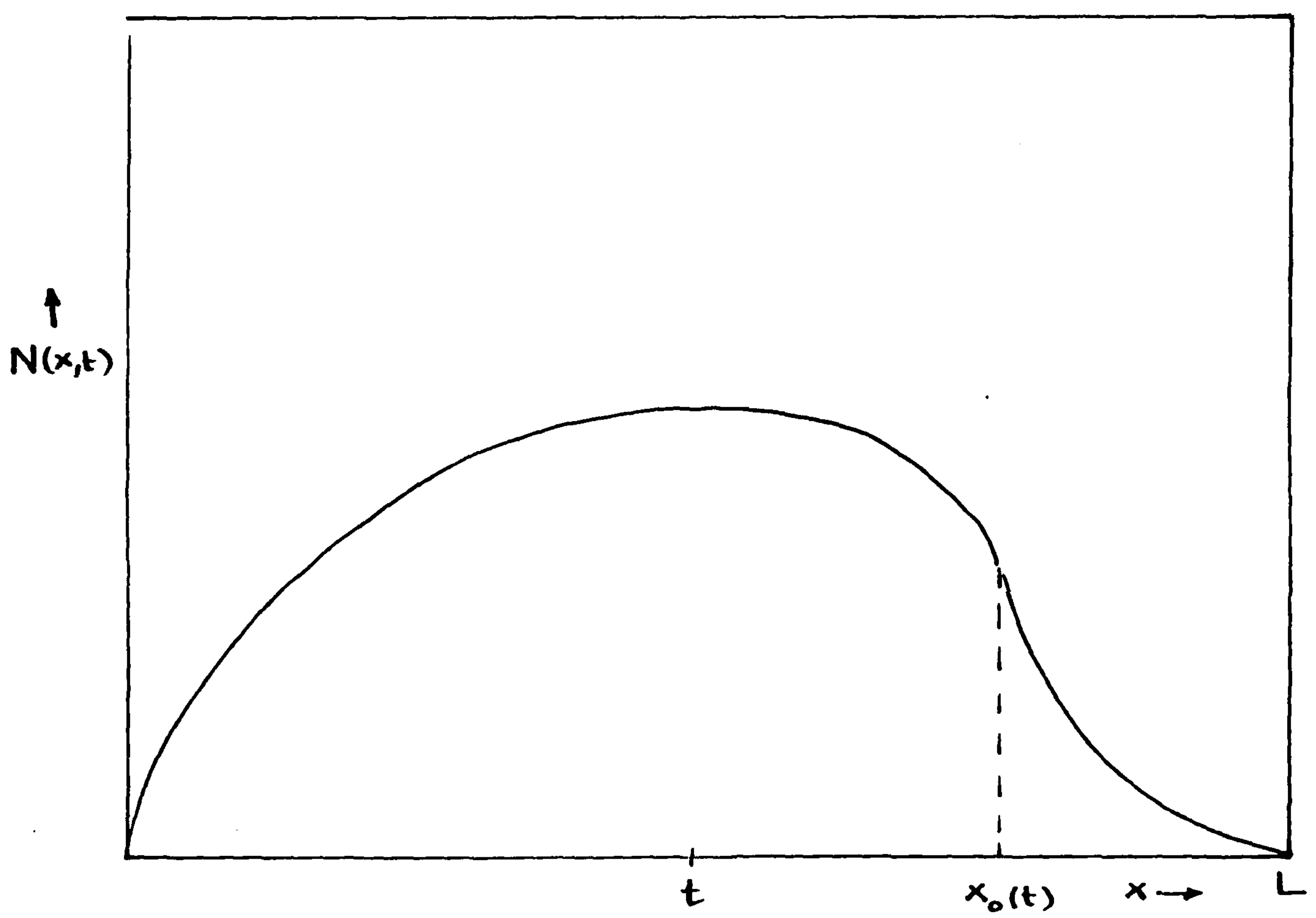


Fig. 1

IMPULSE RESPONSE AND UNIQUENESS OF THE INVERSE PROBLEM

<u>The impulse response</u>: Let us consider the following artificial problem

$$\frac{\partial^2}{\partial x^2} r(x) \frac{\partial^2 \hat{y}}{\partial x^2} = -\rho(x) \frac{\partial \hat{y}}{\partial t}, \quad \begin{array}{l} 0 < x < L \\ t > 0 \end{array} \tag{2.1}$$

with

$$\left.\begin{array}{ll} \frac{\partial^2 \hat{y}}{\partial x^2}(0, t) = \frac{\partial}{\partial x}\left(r(x) \frac{\partial^2 \hat{y}}{\partial x^2}\right)(0, t) & = 0 , \\ \hat{y}(L, t) = \frac{\partial}{\partial x} \hat{y}(L, t) & = 0 , \\ \hat{y}(x, 0) = \eta(x) & \end{array}\right\} \tag{2.2}$$

After making a Laplace transform where

$$y(x, -s) = \int_0^\infty e^{-st} \hat{y}(x, t)\, dt ,$$

we can write (2.1) as follows:

$$(ry'')'' + \rho\, s\, y = \rho\, \eta \tag{2.3}$$

with

$$\left.\begin{array}{llll} y'' = (ry'')' & = 0 & \text{at} & x = 0 \\ y = y' & = 0 & \text{at} & x = L \end{array}\right| \tag{2.4}$$

In order to solve the above boundary value problem, we introduce the appropriate Green's function

$$\left.\begin{aligned} (r\underline{\Omega}'')'' + \rho\, s\, \underline{\Omega} &= \delta(x - z) \\ \underline{\Omega}'' = (r\underline{\Omega}'')' &= 0 \quad \text{at} \quad x = 0 \\ \underline{\Omega} = \underline{\Omega}' &= 0 \quad \text{at} \quad x = L \end{aligned}\right\} \qquad (2.5)$$

which enables us to write

$$y(x, -s) = \int_0^L \underline{\Omega}(x, z; s)\, \rho(z)\, \eta(z)\, dz\ . \qquad (2.6)$$

If we were to replace s by $-\omega^2$, then we would be back to our beam which is here excited by a force $\eta(x)$ applied at time $t = 0$. The case

$$\eta(x) = \delta(x)/\rho(0)$$

would then correspond to a point force applied at the force left end time $t = 0$. By impulse response, we shall denote the measurements of the displacement and slope of the center line at that left end, i.e.

$$\left.\begin{aligned} y(0, -s) &= \underline{\Omega}(0, 0; s) \\ \theta(0, -s) &= \frac{\partial \Omega}{\partial x}(0, 0; s) \end{aligned}\right\} \qquad (2.7)$$

Now, making use of the functions ϕ, ψ, Φ and Ψ introduced previously, we can check that

$$\underline{\Omega}(x, z; -s) = \begin{cases} \dfrac{\phi(x, -s)\, \Phi(z, -s) - \psi(x, -s)\, \Psi(z, -s)}{r^2(0)\, W(0, -s)} & x \leq z \\[2ex] \dfrac{\phi(z, -s)\, \Phi(x, -s) - \psi(z, -s)\, \Psi(x, -s)}{r^2(0)\, W(0, -s)} & x \geq z \end{cases} \qquad (2.8)$$

Consequently

$$y(0, -s) = \frac{1}{r^2(0)} \frac{\Phi(0, -s)}{W(0, -s)}$$

and

$$\theta(0, -s) = -\frac{1}{r^2(0)} \frac{\Psi(0, -s)}{W(0, -s)}$$

or, in view of (1.19), (1.26) and (1.28)

$$y(0, -s) = F_2 \prod_{m=1}^{\infty} \frac{1 + s/\mu_m^2}{1 + s/\omega_m^2} , \tag{2.9a}$$

$$\theta(0, -s) = -F_1 \prod_{m=1}^{\infty} \frac{1 + s/\nu_m^2}{1 + s/\omega_m^2} . \tag{2.9b}$$

The impulse response contains the following information:

(i) the first and second moments F_1 and F_2 of the "flaccidity"

(ii) the three spectra $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ corresponding respectively to the free, non-self adjoint and supported boundary conditions at the left end.

Is the information contained in the impulse response sufficient to characterize a beam uniquely? This is the question we wish to discuss.

Beams with identical spectra: Let us say that the beam characterized by $r(x)$, $\rho(x)$, of length L, clamped at the right end, has spectra $\{\lambda_n\}$, $\{\mu_n\}$, $\{\nu_n\}$, $\{\sigma_n\}$ and $\{\omega_n\}$. There are other distinct beams with the same spectra. Indeed, the two parameter family of beams characterized by

$$\left.\begin{aligned} \tilde{r}(\tilde{x}) &= \alpha\, r\left(\frac{\alpha}{\beta} \right)^{\frac{1}{4}} x) \;, \\ \tilde{\rho}(\tilde{x}) &= \beta\, \rho\left(\frac{x}{\beta} \right)^{\frac{1}{4}} x) \;, \end{aligned}\right\} \qquad (2.10)$$

have the same spectra. (The parameters α and β are positive.) The lengths of these beams are

$$\tilde{L} = \left(\frac{\alpha}{\beta} \right)^{\frac{1}{4}} L \;. \qquad (2.11)$$

The first and second moments of the flaccidity of these beams are:

$$\left.\begin{aligned} \tilde{F}_1 &= \frac{1}{\alpha^{\frac{1}{2}}\, \beta^{\frac{1}{2}}}\, F_1 \\ \tilde{F}_2 &= \frac{1}{\alpha^{\frac{1}{4}}\, \beta^{\frac{3}{4}}}\, F_2 \end{aligned}\right\} \qquad (2.12)$$

If we were to impose the additional requirement that these two moments match those of the beam under consideration, then we would obviously see that $\alpha = \beta = 1$. Hence, the role of F_1 and F_2 is to set the scale of the beam which is being reconstructed.

Pathological Beams with the same impulse response: It is possible to construct two distinct beams with the same impulse response?

Consider a beam $\rho(x)$, $r(x)$ of length L , clamped at the right end. Construct a second beam with the following characteristics:

$$\tilde{\rho}(x) = \begin{cases} \rho(x) & 0 < x < L \\ \bar{\rho} & L < x < \tilde{L} \end{cases} \qquad (2.13a)$$

$$\tilde{r}(x) = \begin{cases} r(x) & 0 < x < L \\ \bar{r} & L < x < \tilde{L} \end{cases} \tag{2.13b}$$

In other words, the second beam is identical to the first one over the interval $(0, L)$ and then has constant properties over the section $(L, \tilde{L})$. Abusing our notations, let us denote by $\phi(x, \omega^2)$, $\psi(x, \omega^2)$ two fundamental solutions of (1.1) which satisfy the boundary conditions at $x = 0$ appropriate for either one of the three configurations entering in the impulse response.

The solution over the interval $(L, \tilde{L})$ is written in terms of functions

$$\cos[(\frac{\bar{\rho}\omega^2}{\bar{r}})^{\frac{1}{4}} (x - \tilde{L})] - \cosh[(\frac{\bar{\rho}\omega^2}{\bar{r}})^{\frac{1}{4}} (x - \tilde{L})]$$

and

$$\sin[(\frac{\bar{\rho}\omega^2}{\bar{r}})^{\frac{1}{4}} (x - \tilde{L})] - \sinh[(\frac{\bar{\rho}\omega^2}{\bar{r}})^{\frac{1}{4}} (x - \tilde{L})]$$

which satisfy the clamped conditions identically.

Piecing the solutions in $(0, L)$ and $(L, \tilde{L})$ by requiring that

$$[\tilde{y}] = [\tilde{y}'] = [\tilde{r}\tilde{y}''] = [(\tilde{r}\tilde{y}'')'] = 0 \tag{2.14}$$

where $[\ ]$ denotes the jump from $L + 0$ to $L - 0$, we obtain the determinantal equations yielding the various spectra, viz.,

$$
0 = \begin{vmatrix}
\phi(L,\omega^2) & \psi(L,\omega^2) & \cos Z - \cosh Z & \sin Z - \sinh Z \\[2ex]
\phi'(L,\omega^2) & \psi'(L,\omega^2) & -\left(\dfrac{\bar{\rho}}{\bar{r}}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}} \{\sin Z + \sinh Z\} & \left(\dfrac{\bar{\rho}}{\bar{r}}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}} \{\cos Z - \cosh Z\} \\[2ex]
(r\phi'')(L,\omega^2) & (r\psi'')(L,\omega^2) & -\bar{r}^{\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}\omega\{\cos Z + \cosh Z\} & -\bar{r}^{\frac{1}{2}}\bar{\rho}^{\frac{1}{2}}\omega\{\sin Z + \sinh Z\} \\[2ex]
(r\phi'')'(L,\omega^2) & (r\psi'')'(L,\omega^2) & \bar{r}^{\frac{1}{4}}\bar{\rho}^{\frac{3}{4}}\omega^{\frac{3}{2}}\{\sin Z - \sinh Z\} & -\bar{r}^{\frac{1}{4}}\bar{\rho}^{\frac{3}{4}}\omega^{\frac{3}{2}}\{\cos Z + \cosh Z\}
\end{vmatrix} \qquad (2.15)
$$

where

$$
Z = \left(\frac{\bar{\rho}}{\bar{r}}\right)^{\frac{1}{4}} \omega^{\frac{1}{2}} (L - \tilde{L}) \ .
$$

Now, we consider the limit

$$\frac{\bar{\rho}}{\bar{r}} \to 0 \tag{2.16a}$$

but such that

$$\bar{r}\,\bar{\rho} = \alpha \ . \tag{2.16b}$$

This means, that in the limit the added section is made up of a weightless rod of infinite rigidity. Then, dividing the last column of (2.15) by $\bar{r}^{1/4}\,\bar{\rho}^{3/4}$ and taking the limit mentioned above, we note that (2.15) becomes:

$$\begin{vmatrix} \phi & \psi & 0 & 0 \\ \phi' & \psi' & 0 & 0 \\ r\phi'' & r\psi' & -2\alpha\omega & -\omega \\ (r\phi'')' & (r\psi'')' & 0 & -2\omega^{\frac{3}{2}} \end{vmatrix} = 0$$

or better still

$$\begin{vmatrix} \phi & \psi \\ \phi' & \psi' \end{vmatrix} = 0 \ .$$

But this is the generic determinal equation for the smaller beam! In addition, since the flaccidity of the second beam is zero over the

additional interval $(L, \tilde{L})$

$$\tilde{F}_1 = F_1$$

and

$$\tilde{F}_2 = F_2$$

Thus, for the clamped right end case, given any beam with impulse response $\{\lambda_n\}$, $\{\nu_n\}$, $\{\omega_n\}$, F_1 and F_2, we can construct a class of beams with the same impulse response, by simply tacking onto the original beam, an infinitely rigid weightless rod of arbitrary length.

This result is easy to understand physically: the section $(L, \tilde{L})$, because of its very nature, is never set in motion by the initial excitation and hence, never felt.

The question remains: are there other types of less pathological beams which have the same impulse response? The answer is no.

Toward the Uniqueness Theorem: We assume that two beams $\rho^{(1)}$, $r^{(1)}$ and $\rho^{(2)}$, $r^{(2)}$ have the same impulse response. Since the information about the length of the beams is not part of the impulse response, we must assume that their lengths, $L^{(1)}$ and $L^{(2)}$, are not necessary equals. This presents a first obstacle to the adaptation of Levinson's (1949) classical proof of the uniqueness of the inverse Strum-Lionville to the problem at hand. Indeed, in order to compare the two beams we must bring them to some common ground. If $x^{(1)}$ and $x^{(2)}$ are coordinates suitable for each beam, then, we can accomplish our aim by introducing a new variable ξ as follows:

$$\xi = \int_0^{x^{(1)}} \left[\frac{\rho^{(1)}}{r^{(1)}} \right]^{\frac{1}{4}} dx \tag{2.17a}$$

and

$$\xi = \int_0^{x^{(2)}} \left[\frac{\rho^{(2)}}{r^{(2)}} \right]^{\frac{1}{4}} dx \ . \tag{2.17b}$$

Note that as $x^{(1)}$ and $x^{(2)}$ vary from 0 to respectively $L^{(1)}$ and $L^{(2)}$, ξ varies from 0 to Ξ which was defined in (1.39). This is a consequence of the fact that eigenvalues of the two beams have the same asymptotic behavior.

The functions $\xi(x^{(1)})$ and $\xi(x^{(2)})$ defined in (2.17) are nondecreasing, and hence are invertible:

$$\left. \begin{aligned} x^{(1)} &= x^{(1)}(\xi) \\ x^{(2)} &= x^{(2)}(\xi) \end{aligned} \right\} \tag{2.18}$$

We are now ready to define the following strange integral of a hybrid version of the Green's function $\underline{\Omega}$ given in (2.8), namely

$$\mathcal{J}(\xi) = \frac{1}{2\pi i} \int_{-i\infty}^{+i\infty} ds \int_0^{\Xi} \rho^{(2)}(x^{(2)}(\xi)) \, \underline{\underline{\Omega}}^{(2,1)}(x^{(2)}(\zeta), x^{(1)}(\xi); -s) \, h(\zeta) \frac{dx^{(2)}}{d\zeta} \, d\zeta$$

... (2.19)

where

$$\underline{\underline{\Omega}}^{(2,1)}(x^{(2)}(\zeta),\ x^{(1)}(\xi);\ -s)$$

$$= \begin{cases} \dfrac{\phi^{(2)}(x^{(2)}(\zeta),-s)\Phi^{(1)}(x^{(1)}(\xi),-s)-\psi^{(2)}(x^{(2)}(\zeta),-s)\Psi^{(1)}(x^{(1)}(\xi),-s)}{r^2(0)W^{(1)}(0,-s)} & \text{for } x^{(2)} \le x^{(1)} \\[2ex] \dfrac{\phi^{(1)}(x^{(1)}(\xi),-s)\Phi^{(2)}(x^{(2)}(\zeta),-s)-\psi^{(1)}(x^{(1)}(\xi),-s)\Psi^{(2)}(x^{(2)}(\zeta),-s)}{r^2(0)W^{(2)}(0,-s)} & \text{for } x^{(1)} \le x^{(2)} \end{cases}$$

$$\ldots \qquad (2.20)$$

and $h(\zeta)$ is an arbitrary function.

We digress at this stage to explain why $r(0)$ enters in the definition of $\underline{\underline{\Omega}}^{(2,1)}$ without any superscript. This is due to the fact that

$$r^{(1)}(0) = r^{(2)}(0) \qquad (2.21)$$

This result , as well as

$$\rho^{(1)}(0) = \rho^{(2)}(0) \qquad (2.22)$$

will be derived next.

<u>End-Point Identities</u>: The above results are obtained by confronting the product representation (1.26) of $\phi^{(i)}(0,\omega^2)$ with its asymptotic representation (1.24a). Indeed, this implies that

$$[r^{(i)}(L)^{(i)}]^2 \; F_2 \prod_{m=1}^{\infty} (1 + \frac{s}{\mu_m^2}) \sim \frac{1}{2\sqrt{2}\, s^{\frac{3}{4}}} \; \frac{[r^{(i)}(L^{(i)})]^{\frac{3}{2}}}{[\rho^{(i)}(L^{(i)})]^{\frac{1}{2}}} \cdot \left[\frac{r^{(i)}(0)}{\rho^{(i)}(0)}\right]^{\frac{1}{4}} \cdot$$

$$\cdot \{\sinh \sqrt{2}\, s^{\frac{1}{4}}\, \Xi - \sin \sqrt{2}\, s^{\frac{1}{4}}\, \Xi\} \tag{2.23}$$

where $i = 1, 2$. Consequently

$$\frac{1}{[r^{(1)}(L^{(1)})]^{\frac{1}{2}}} \cdot \frac{1}{[\rho^{(1)}(L^{(1)})]^{\frac{1}{2}}} \cdot \left[\frac{r^{(1)}(0)}{\rho^{(1)}(0)}\right]^{\frac{1}{4}}$$

$$= \frac{1}{[r^{(2)}(L^{(2)})]^{\frac{1}{2}}} \quad \frac{1}{[\rho^{(2)}(L^{(2)})]^{\frac{1}{2}}} \left[\frac{r^{(2)}(0)}{\rho^{(2)}(0)}\right]^{\frac{1}{4}} \tag{2.24}$$

The same procedure applied to $\psi^{(i)}(0, \omega^2)$, viz

$$[r^{(i)}(L^{(i)})]^2 \; F_1 \prod_{m=1}^{\infty} (1 + \frac{s}{\nu_m^2}) \sim \frac{1}{2\, s^{\frac{1}{2}}} \; \frac{[r^{(i)}(L^{(i)})]^{\frac{3}{2}}}{[\rho^{(i)}(L^{(i)})]^{\frac{1}{2}}} \quad .$$

$$\cdot \{\cosh^2(\frac{s^{\frac{1}{4}}}{\sqrt{2}}\, \Xi) - \cos^2(\frac{s^{\frac{1}{4}}}{\sqrt{2}}\, \Xi)\} \; , \tag{2.25}$$

yields

$$\frac{1}{[r^{(1)}(L^{(1)})]^{\frac{1}{2}}} \cdot \frac{1}{[\rho^{(1)}(L^{(1)})]^{\frac{1}{2}}} = \frac{1}{[r^{(2)}(L^{(2)})]^{\frac{1}{2}}} \cdot \frac{1}{[\rho^{(2)}(L^{(2)})]^{\frac{1}{2}}} \tag{2.26}$$

In the same vein, we can confront the product representation (1.19) for $W(0, -s)$ with its asymptotic representation (1.26), viz.

$$[r^{(i)}(L^{(i)})]^2 \prod_{m=1}^{\infty} (1 + \frac{s}{\omega_m^2}) \sim \tfrac{1}{2} \left[\frac{\rho^{(i)}(0)\, r^{(i)}(0)}{\rho^{(i)}(L^{(i)})} \right]^{\frac{1}{2}} [r^{(i)}(L^{(i)})]^{\frac{3}{2}}$$

$$\cdot \{\cosh^2 \frac{s^{\frac{1}{4}}}{\sqrt{2}} \Xi + \omega s^2 \frac{s^{\frac{1}{4}}}{\sqrt{2}} \Xi\} \ , \qquad (2.27)$$

from which we deduce that

$$\left[\frac{r^{(1)}(0)}{r^{(1)}(L^{(1)})}\right]^{\frac{1}{2}} \left[\frac{\rho^{(1)}(0)}{\rho^{(1)}(L^{(1)})}\right]^{\frac{1}{2}} = \left[\frac{r^{(2)}(0)}{r^{(2)}(L^{(2)})}\right]^{\frac{1}{2}} \left[\frac{\rho^{(2)}(0)}{\rho^{(2)}(L^{(2)})}\right]^{\frac{1}{2}} . \qquad (2.28)$$

Making use of (2.26) we can rewrite (2.24) and (2.28) as follows

$$\frac{r^{(1)}(0)}{\rho^{(1)}(0)} = \frac{r^{(2)}(0)}{\rho^{(2)}(0)} \ ,$$

$$r^{(1)}(0)\ \rho^{(1)}(0) = r^{(2)}(0)\ \rho^{(2)}(0) \ ,$$

which imply (2.21) and (2.22). Therefore, if the two beams clamped at the right end have the same impulse response, their densities and rigidities must coincide at the left end. Therefore, we are justified in omitting the superscripts.

In addition, they must satisfy (2.26), i.e.

$$r^{(1)}(L^{(1)})\ \rho^{(1)}(L^{(1)}) = r^{(2)}(L^{(2)})\ \rho^{(2)}(L^{(2)}) \ .$$

Note that the above product is finite even for the pathological beams previously discussed.

Evaluation of $\mathcal{J}(\xi)$ by Calculus of Residues: An examination of the integrand of $\mathcal{J}(\xi)$ reveals that the path of integration can be closed by a half-circle in the left half of the s-plane. The only singularities are simple poles at $s = -\omega_n^2$ and a straight-forward application of the calculus of residues yields:

$$
\begin{aligned}
\mathcal{J}(\xi) = \sum_{n=1}^{\infty} \Bigg[& \Big\{ \Phi_n^{(1)}(x^{(1)}) \int_0^{\xi} \rho^{(2)}(x^{(2)})\, \phi^{(2)}(x^{(2)}, \omega_n^2) h \frac{dx^{(2)}}{d\zeta}\, d\zeta \\
& - \Psi_n^{(1)}(x^{(1)}) \int_0^{\xi} \rho^{(2)}(x^{(2)})\, \psi^{(2)}(x^{(2)}, \omega^2) h \frac{dx^{(2)}}{d\zeta}\, d\zeta \Big\} \frac{\omega_n^2}{[r^{(1)}(L^{(1)})]^2 \prod\limits_{m \neq n} (1 - \frac{\omega_n^2}{\omega^2})} \\
& + \Big\{ \phi^{(1)}(x^{(1)}, \omega_n^2) \int_{\xi}^{\Xi} \rho^{(2)}(x^{(2)})\, \Phi_n^{(2)}(x^{(2)}) h \frac{dx^{(2)}}{d\zeta}\, d\zeta \\
& - \psi^{(1)}(x^{(1)}, \omega_n^2) \int_{\xi}^{\Xi} \rho^{(2)}(x^{(2)})\, \Psi_n^{(2)}(x^{(2)}) h \frac{dx^{(2)}}{d\zeta}\, d\zeta \Big\} \frac{\omega_n^2}{[r^{(2)}(L^{(2)})]^2 \prod\limits_{m \neq n} (1 - \frac{\omega_n^2}{\omega_m^2})} \Bigg]
\end{aligned}
\qquad \ldots (2.29)
$$

In the above formula, we have not bothered to write explicitly that $x^{(1)}$ and $x^{(2)}$ are functions of ξ and ζ. We have also used the representation (1.19) for the Wronskians.

At this stage, we make use of the fact that $\Phi_n^{(i)}(x^{(i)})$ and $\Psi_n^{(i)}(x^{(i)})$ for $i = 1, 2$ are related by a constant of proportionality k_n, which, as (1.32) reveals, is *soley dependent on the impulse response data*, and hence identical for both beams. As a result (2.29) becomes

$$\mathfrak{J}(\xi) = \sum_{n=1}^{\infty} \frac{\omega_n^2}{\prod\limits_{m\neq n} \left(1 - \frac{\omega_n^2}{\omega_m^2}\right)} \Bigg[[r^{(1)}(L^{(1)})]^{-2} \left\{ \int_0^{\xi} \rho^{(2)}(x^{(2)})[k_n \, \phi^{(2)}(x^{(2)}, \omega_n^2)\right.$$

$$\left. - \, \psi^{(2)}(x^{(2)}, \omega_n^2)] h \, \frac{dx^{(2)}}{d\zeta} \, d\zeta \right\} \Psi_n^{(1)}(x^{(1)})$$

$$+ \, [r^{(2)}(L^{(2)})]^{-2} \left\{ \int_{\xi}^{\Xi} \rho^{(2)}(x^{(2)}) \, \Psi_n^{(2)}(x^{(2)}) h \, \frac{dx^{(2)}}{d\zeta} \, d\zeta \right\} \cdot$$

$$[k_n \, \phi^{(1)}(x^{(1)}, \omega_n^2) - \psi^{(1)}(x^{(1)}, \omega_n^2)] \Bigg]$$

The above expression can be further simplified by means of (1.34'), i.e. by recalling that

$$k_n \, \phi^{(i)}(x^{(i)}, \omega_n^2) - \psi^{(i)}(x^{(i)}, \omega_n^2) = k_n^{(i)*} \, \Psi_n^{(i)}(x^{(i)})$$

where, as (1.36) indicates

$$[r^{(1)}(L^{(1)})]^2 \, k_n^{(1)*} = [r^{(2)}(L^{(2)})]^2 \, k_n^{(2)*}$$

$$= \frac{k_n}{F_1 \prod\limits_m \left(1 - \frac{\omega_n^2}{\nu_m^2}\right)} \, .$$

As a result

$$\mathcal{J}(\xi) = \sum_{n=1}^{\infty} \frac{1}{[r^{(1)}(L^{(1)})r^{(2)}(L^{(2)})]^2} \cdot \frac{\omega_n^2 k_n}{F_1} \cdot$$

$$\frac{\int_0^{\Xi} \rho^{(2)}(x^{(2)}) \, \Psi_n^{(2)}(x^{(2)}) h \, \frac{dx^{(2)}}{d\zeta} \, d\zeta}{\prod\limits_{m \neq n} (1 - \frac{\omega_n^2}{\omega_m^2}) \prod\limits_{m} (1 - \frac{\omega_n^2}{\nu_m^2})} \cdot \Psi_n^{(1)}(x^{(1)}) \qquad (2.30)$$

To avoid problems in the pathological cases, it is preferable to free the above expression of the terms $r^{(1)}(L^{(1)})$ $r^{(2)}(L^{(2)})$. This can be done by defining the norm of $\Psi_n^{(i)}$, viz.

$$||\Psi_n^{(i)}||^2 = \int_0^{L^{(i)}} \rho^{(i)}(x^{(i)})[\Psi_n^{(i)}(x^{(i)})]^2 \, dx^{(i)} . \qquad (2.31)$$

The superscripts are not necessary for the actual evaluation of the above norms; hence they shall be omitted temporarily. We recall the equations for $\Phi(x, -s)$ and $\Psi_n(x)$, viz.

$$\frac{d^2}{dx^2} r \frac{d^2\Phi}{dx^2} = -s \rho \Phi ,$$

and

$$\frac{d^2}{dx^2} r \frac{d^2 \Psi_n}{dx^2} = - \omega_n^2 \rho \Psi_n .$$

Multiplying the first by Ψ_n, the second by Φ, subtracting, integrating the resulting expression over $(0, L)$, we get after making use of

(1.22) and (1.23)

$$\Psi_n(0)\ r(0)\ \Phi'''(0,\ s) = \omega_n^2\left(1 + \frac{s}{\omega_n^2}\right)\int_0^L \rho\ \Phi\ \Psi_n\ dx\ . \tag{2.32}$$

The product representations (1.28) and (1.30b), enable us to write:

$$\frac{r^4(L)\ F_1}{\omega_n^2}\ \prod_{m\neq n}\left(1 + \frac{s}{\omega_m^2}\right)\prod_m\left(1 - \frac{\omega_n^2}{\nu_m^2}\right) = \int_0^L \rho\ \Phi\ \Psi_n\ dx\ .$$

It only remains now to let $s \to -\omega_n^2$ to see that

$$\frac{r^4(L)\ F_1}{\omega_n^2}\ \prod_{m\neq n}\left(1 - \frac{\omega_n^2}{\omega_m^2}\right)\prod\left(1 - \frac{\omega_n^2}{\nu_m^2}\right) = k_n\ ||\Phi_n||^2 \tag{2.33}$$

and consequently (2.30) becomes

$$\mathfrak{J}(\xi) = \sum_{n=1}^{\infty} \frac{\int_0^{\Xi} \rho^{(2)}(x^{(2)})\ \Psi_n^{(2)}(x^{(2)})h\ \frac{dx}{d\zeta}^{(2)}\ d\zeta}{||\Psi_n^{(1)}||\ ||\Psi_n^{(2)}||}\ \Psi_n^{(1)}(x^{(1)})\ . \tag{2.34}$$

Needless to say, if the two beams were identical, then we could drop the superscripts altogether and get the classical eigenfunction expression

$$\mathfrak{J}(\xi(x)) = \sum_{n=1}^{\infty} \frac{\int_0^L \rho\ \Psi_n\ h(\zeta(x))\ dx}{||\Psi_n||^2}\ \Psi_n(x)\ ,$$

as a special case of that more general eigenfunction expansion. Incidentally, in this particular case, $\mathcal{J}(\xi)$ is just $h(\xi)$.

Evaluation of $\mathcal{J}(\xi)$ via residue of pole at infinity: An alternative expression for $\mathcal{J}(\xi)$ is obtained by taking the contour in the s-plane to be a very large circle and replacing $\underline{\underline{\Omega}}^{(2,\,1)}(x^{(2)},\, x^{(1)};\, -s)$ by its asymptotic expression for large $|s|$. The calculation is straightforward by tedious. Once this is done, we perform the integration over ξ-from $(0,\, \xi)$ and $(\xi,\, \Xi)$ asymptotically: this requires a mere integration by parts, provided, of course, that $r,\, \rho$ are differentiable. The remaining integral over s is now trivial since to leading order the s-dependence of the integrand is $\frac{1}{s}$. The upshot of these calculations is

$$\mathcal{J}(\xi) = \left[\frac{r^{(2)}(x^{(2)}(\xi))}{r^{(1)}(x^{(1)}(\xi))} \right]^{\frac{1}{8}} \left[\frac{\rho^{(2)}(x^{(2)}(\xi))}{\rho^{(1)}(x^{(1)}(\xi))} \right]^{\frac{3}{8}} h(\xi) \tag{2.35}$$

Equating the two expressions for $\mathcal{J}(\xi)$ we get:

$$\left[\frac{r^{(2)}(x^{(2)})}{r^{(1)}(x^{(1)})} \right]^{\frac{1}{8}} \left[\frac{\rho^{(2)}(x^{(2)})}{\rho^{(1)}(x^{(1)})} \right]^{\frac{3}{8}} h = \sum \frac{\int_0^{\Xi} \rho^{(2)}\, \Psi_n^{(2)}\, h\, \frac{dx^{(2)}}{d\zeta}\, d\zeta}{||\Psi_n^{(1)}||\ ||\Psi_n^{(2)}||}\, \Psi_n^{(1)}(x^{(1)}) \qquad \ldots \tag{2.36}$$

But the left hand side can also be written in terms of a standard eigenfunction expansion, viz.

$$\left[\frac{r^{(2)}(x^{(2)})}{r^{(1)}(x^{(1)})}\right]^{\frac{1}{8}}\left[\frac{\rho^{(2)}(x^{(2)})}{\rho^{(1)}(x^{(1)})}\right]^{\frac{3}{8}} h = \sum \frac{\int_0^{\Xi} \rho^{(1)}\psi_n^{(1)}\left[\frac{r^{(2)}}{r^{(1)}}\right]^{\frac{1}{8}}\left[\frac{\rho^{(2)}}{\rho^{(1)}}\right]^{\frac{3}{8}} h \frac{dx^{(1)}}{d\zeta}\, d\zeta}{||\psi_n^{(1)}||^2} \psi_n^{(1)}(x^{(1)})$$

... (2.37)

from which it follows that

$$||\psi_n^{(2)}||^{-1} \int_0^{\Xi} \rho^{(2)}\, \psi_n^{(2)}\, h\, \frac{dx^{(2)}}{d\zeta}\, d\zeta = ||\psi_n^{(1)}||^{-1} \int_0^{\Xi} \rho^{(1)}\, \psi_n^{(1)} \left[\frac{r^{(2)}}{r^{(1)}}\right]^{\frac{1}{8}} \left[\frac{\rho^{(2)}}{\rho^{(1)}}\right]^{\frac{3}{8}} h \, \frac{dx^{(1)}}{d\zeta}\, d\zeta$$

for $n = 1, 2, \ldots$

... (2.38)

and since the function h is completely arbitrary

$$[\rho^{(2)}(x^{(2)}(\xi))]^{\frac{3}{8}}\, [r^{(2)}(x^{(2)}(\xi))]^{\frac{1}{8}} \quad \frac{\psi_n^{(2)}(x^{(2)}(\xi))}{||\psi_n^{(2)}||}$$

$$= [\rho^{(1)}(x^{(1)}(\xi))]^{\frac{3}{8}}\, [r^{(1)}(x^{(1)}(\xi))]^{\frac{1}{8}} \quad \frac{\psi_n^{(1)}(x^{(1)}(\xi))}{||\psi_n^{(1)}||} \; .$$

for $n = 1, 2, \ldots$

... (2.39)

We must again digress before we can draw any conclusion from (2.39).

The Liouville transformation and the canonical 4^{th} order equation: Willy nilly, we have been pushed first into the introduction of the variable

$$\xi = \int_0^x \left(\frac{\rho}{r} \right)^{1/4} dx$$

and now into considering the functions

$$U_n(\xi) = [\rho(x(\xi))]^{\frac{3}{8}} \; [r(x(\xi))]^{\frac{1}{8}} \; \frac{\Psi_n(x(\xi))}{||\Psi_n||} \; . \qquad (2.40)$$

We can think of these new variables as constituting a Liouville transformation. In terms of them, the original eigenvalue problem, viz.

$$\left.\begin{array}{lll} \dfrac{d^2}{dx^2} \; r \, \dfrac{d^2 \Psi_n}{dx^2} = \omega_n^2 \, \rho \, \Psi_n & & \\ \Psi_n'' = (r \, \Psi_n'')' = 0 \; , & \text{at} & x = 0 \; , \\ \Psi_n = \Psi_n' = 0 \; , & \text{at} & x = L \; , \end{array}\right\}$$

is transformed into

$$\frac{d^4 U_n}{d\xi^4} + \frac{d}{d\xi} \; A(\xi) \, \frac{d \, U_n}{d\xi} + B(\xi) \, U_n = \omega_n^2 \, U_n \qquad (2.41)$$

with

$$\left.\begin{array}{l} \dfrac{d^2 U_n}{d\xi^2} + a \, \dfrac{d \, U_n}{d\xi} + b \, U_n = 0 \\ \dfrac{d^3 U_n}{d\xi^3} + c \, \dfrac{d^2 U_n}{d\xi^2} + d \, \dfrac{d \, U_n}{d\xi} + e \, U_n = 0 \end{array}\right\} \text{at } \xi = 0 \quad (2.42a)$$

and

$$U_n = \frac{d\,U_n}{d\xi} = 0 \qquad \text{at} \quad \xi = \Xi \tag{2.42b}$$

The coefficients $A(\xi)$ and $B(\xi)$ are related to ρ and r thus:

$$A = 4\,\frac{q_{\xi\xi}}{q} - 6\,\frac{q_\xi^2}{q^2} - 2\,\frac{q_\xi p_\xi}{qp} + \frac{p_{\xi\xi}}{p} - 2\,\frac{p_\xi^2}{p^2} \tag{2.43a}$$

$$B = \frac{q_{\xi\xi\xi\xi}}{q} - 4\,\frac{q_\xi q_{\xi\xi\xi}}{q^2} - 2\,\frac{q_{\xi\xi}^2}{q^2} + 6\,\frac{q_\xi^2 q_{\xi\xi}}{q^3}$$

$$+ \frac{q_\xi p_{\xi\xi\xi}}{qp} + \frac{q_{\xi\xi} p_{\xi\xi}}{qp} - 3\,\frac{q_\xi^2 p_{\xi\xi}}{q^2 p} - \frac{q_\xi p_\xi p_{\xi\xi}}{qp^2}$$

$$- 4\,\frac{q_\xi q_{\xi\xi} p_\xi}{q^2 p} + 6\,\frac{q_\xi^3 p_\xi}{q^3 p} - 2\,\frac{q_\xi^2 p_\xi^2}{q^2 p^2} \tag{2.43b}$$

where

$$p(\xi) = \left(\frac{\rho(x)}{r(x)} \right)^{\frac{1}{4}} , \tag{2.44}$$

and

$$q(\xi) = \rho^{-\frac{3}{8}}(x)\; r^{-\frac{1}{8}}(x) \; . \tag{2.45}$$

Also

$$\begin{aligned}
a &= \frac{2q_\xi}{q} + \frac{p_\xi}{p} , \\
b &= \frac{q_{\xi\xi}}{q} + \frac{q_\xi p_\xi}{qp} , \\
c &= \frac{q_\xi}{q} , \\
d &= 3\,\frac{q_{\xi\xi}}{q} - 4\,\frac{q_\xi^2}{q^2} - 2\,\frac{q_\xi p_\xi}{qp} + \frac{p_{\xi\xi}}{p} - 2\,\frac{p_\xi^2}{p^2} , \\
e &= \frac{q_{\xi\xi\xi}}{q} - \frac{q_\xi q_{\xi\xi}}{q^2} + \frac{p_{\xi\xi} q_\xi}{pq} - 2\,\frac{p_\xi q_\xi^2}{pq^2} \qquad \frac{p_\xi^2 q_\xi}{p^2 q}
\end{aligned} \tag{2.46}$$

Since we have been led to a consideration of the canonical 4^{th} order equation (2.41), one may ask why did we not consider this equation from the beginning? As a matter of fact, I have considered the inverse eigenvalue problem for this equation in a previous paper (Barcilon, 1974). That analysis was valid only in the case where the eigenvalues were simple. This assumption is correct provided that the canonical equation can be written as

$$\alpha_0 \frac{d}{dx} \alpha_1 \frac{d}{dx} \alpha_2 \frac{d}{dx} \alpha_3 \frac{d}{dx} \alpha_4 \, U = \omega^2 U \qquad (2.47)$$

where α_0, α_1, α_2, α_3 and α_4 are positive and if the boundary conditions satisfy certain requirements (Karlin, 1971). Thus, one is not dealing with the most general canonical 4^{th} order equation, but with that subclass which has simple eigenvalues. Under those circumstances, it is preferrable to deal with the beam equation.

Before resuming our discussion about uniqueness, we shall introduce some new variables which will simplify the formulas for $A(\xi)$, $B(\xi)$, a, ..., e, namely

$$\left.\begin{aligned} P(\xi) &= \frac{\rho_\xi}{\rho} \, , \\ Q(\xi) &= \frac{q_\xi}{q} \, , \end{aligned}\right\} \qquad (2.48)$$

or equivalently

$$\left.\begin{aligned} \rho(\xi) &= \rho(0) \exp \left(\int_0^\xi P(\zeta) \, d\zeta \right) , \\ q(\xi) &= q(0) \exp \left(\int_0^\xi Q(\zeta) \, d\zeta \right) . \end{aligned}\right\} \qquad (2.48')$$

With these variables, (2.43) becomes

$$4Q_{\xi} + P_{\xi} - 2Q^2 - 2QP - P^2 - A = 0 \tag{2.49a}$$

and

$$\begin{aligned} Q_{\xi\xi\xi} &+ QP_{\xi\xi} + Q_{\xi}P_{\xi} - 4Q^2Q_{\xi} + Q_{\xi}^2 \\ &- 4QQ_{\xi}P + Q_{\xi}P^2 + 2QPP_{\xi} - 2Q^2P_{\xi} \\ &+ Q^4 + 2Q^3P - 4Q^2P^2 - B = 0 \end{aligned} \tag{2.49b}$$

or better still

$$\underset{\sim}{Q}_{\xi} = \underset{\sim}{F}(Q) + \underset{\sim}{A} \tag{2.50}$$

where

$$\underset{\sim}{Q} = \begin{bmatrix} Q \\ P \\ Q_{\xi} \\ P_{\xi} \\ Q_{\xi\xi} \end{bmatrix} , \tag{2.51}$$

$$\underset{\sim}{F} = \begin{bmatrix} Q_{\xi} \\ P_{\xi} \\ Q_{\xi\xi} \\ -4Q_{\xi\xi} + 4QQ_{\xi} + 2Q_{\xi}P + 2QP_{\xi} + 2PP_{\xi} \\ +4QQ_{\xi\xi} - Q_{\xi}P_{\xi} - Q_{\xi}^2 + 2QQ_{\xi}P - 4QPP_{\xi} \\ - Q_{\xi}P + Q^4 + 2Q^3P - 4Q^2P^2 \end{bmatrix} \tag{2.52}$$

and

$$\underset{\sim}{A} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ A_\xi \\ B - QA_\xi \end{bmatrix} \tag{2.53}$$

Finally, the equations (2.46) can be written as follows

$$\underset{\sim}{Q}(0) = \begin{bmatrix} c \\ a - 2c \\ b + c^2 - ac \\ d - 3b - 2c^2 + ac \\ e + cb - ac^2 - cd + 2c^3 \end{bmatrix} \tag{2.54}$$

Uniqueness results: Let us summarize the current state of affairs. Given two beams $\rho^{(1)}(x^{(1)})$, $r^{(1)}(x^{(1)})$ and $\rho^{(2)}(x^{(2)})$, $r^{(2)}(x^{(2)})$ with the same impulse response, then, provided that these functions are sufficiently smooth, we can make a Liouville transformation and write

$$\frac{d^4U_n}{d\xi^4} + \frac{d}{d\xi} \; A^{(1)} \frac{dU_n}{d\xi} + B^{(1)} \; U_n = \omega_n^2 \; U_n \tag{2.55a}$$

and

$$\frac{d^4U_n}{d\xi^4} + \frac{d}{d\xi} \; A^{(2)} \frac{d\;U_n}{d\xi} + B^{(2)} \; U_n = \omega_n^2 \; U_n \; . \tag{2.55b}$$

The fact that U_n has no superscript is a consequence of (2.39), viz.

$$\frac{\Psi_n^{(1)}(x^{(1)})}{q^{(1)}(\xi)\ ||\Psi_n^{(1)}||} = U_n(\xi) = \frac{\Psi_n^{(2)}(x^{(2)})}{q^{(2)}(\xi)\ ||\Psi_n^{(2)}||} \qquad \text{for every } n.$$

The coefficients $A^{(1)}$, $B^{(1)}$ and $A^{(2)}$, $B^{(2)}$ are obtained by substituting $p^{(1)}$, $q^{(1)}$ and $p^{(2)}$, $q^{(2)}$ in (2.43). Now, if we were to subtract (2.55b) from (2.55a) we would get

$$\frac{d}{d\xi}\,(A^{(1)} - A^{(2)})\,\frac{dU_n}{d\xi} + (B^{(1)} - B^{(2)})\,U_n = 0\ . \tag{2.56}$$

Since U_n satisfies (2.42b), we are forced to conclude that

$$\left.\begin{aligned} A^{(1)}(\xi) &= A^{(2)}(\xi) = A(\xi)\ , \\ B^{(1)}(\xi) &= B^{(2)}(\xi) = B(\xi)\ . \end{aligned}\right\} \tag{2.57}$$

or else, U_n would be identically zero. In a similar way, a consideration of the boundary conditions (2.42a) at $\xi = 0$, implies that

$$\left.\begin{aligned} a^{(1)} &= a^{(2)} = a\ , \\ b^{(1)} &= b^{(2)} = b\ , \\ c^{(1)} &= c^{(2)} = c\ , \\ d^{(1)} &= d^{(2)} = d\ , \\ e^{(1)} &= e^{(2)} = e\ . \end{aligned}\right\} \tag{2.58}$$

(This can be seen, for instance, by considering eigenfunctions associated with large values of n .) As a result, the vectors $\underset{\sim}{Q}^{(1)}$ and $\underset{\sim}{Q}^{(2)}$ are solutions of the same differential equation (2.50) with initial conditions

(2.54). Assuming that the solution of this initial value problem exits, it is unique since F is Lipchitz. Consequently

$$\underset{\sim}{Q}^{(1)}(\xi) = \underset{\sim}{Q}^{(2)}(\xi)$$

and in particular

$$\left.\begin{aligned} Q^{(1)}(\xi) &= Q^{(2)}(\xi) , \\ P^{(1)}(\xi) &= P^{(2)}(\xi) . \end{aligned}\right\} \tag{2.59}$$

This, in turn, implies that

$$\frac{p^{(1)}(\xi)}{p^{(1)}(0)} = \frac{p^{(2)}(\xi)}{p^{(2)}(0)}$$

and

$$\frac{q^{(1)}(\xi)}{q^{(1)}(0)} \qquad \frac{q^{(2)}(\xi)}{q^{(2)}(0)} ,$$

and finally, in view of (2.21) - (2.22)

$$\left.\begin{aligned} \rho^{(1)}(x^{(1)}(\xi)) &= \rho^{(2)}(x^{(2)}(\xi)) , \\ r^{(1)}(x^{(1)}(\xi)) &= r^{(2)}(x^{(2)}(\xi)) . \end{aligned}\right\} \tag{2.60}$$

For the smooth class of functions which we have been considering, we can in fact go further and see that $L^{(1)} = L^{(2)}$, and hence

$$\left.\begin{aligned} \rho^{(1)}(x) &= \rho^{(2)}(x) , \\ r^{(1)}(x) &= r^{(2)}(x) . \end{aligned}\right\} \qquad (2.61)$$

However, because of the pathological beams previously discussed, I suspect that (2.60) is the best result possible. Thus, if we are willing to disregard the pathological beams, we can say that the impulse response determines a beam uniquely.

EXISTENCE AND CONSTRUCTION OF SOLUTION

In this section we shall outline two procedures for reconstructing a beam from its impulse response. One of these procedures was previously presented by Barcilon (1979a, 1979b). The central idea revolves around the use of continued fractions and owes much to the work of Krein (1951, 1952a).

The procedure for reconstructing $\rho(x)$ and $r(x)$, given F_1, F_2 and $\{\omega_n, \nu_n, \mu_n\}_1^\infty$, will work provided that the data are *bona fide* data. Thus, we shall have to elaborate criteria for recognizing whether or not a solution exists for a given impulse response. We do not have definitive results about all such criteria: we shall derive a few and suggest the need for more.

Before embarking on this program, it might be useful to review the situation for the inverse problem for a vibrating string. The impulse response for a string in the free/fixed configuration consists of (i) the length L of the string and (ii) the spectra $\{\lambda_n\}_1^\infty$ and $\{\mu_n\}_1^\infty$ associated with the fixed/fixed and free/fixed configurations. The numbers L, $\{\mu_n, \lambda_n\}_1^\infty$ constitute a *bona fide* impulse response if they satisfy the following conditions:

(i) asymptotic behavior

$$\lambda_n \sim \mu_n \sim \frac{n\pi}{\int_0^L \rho^{1/2}(x)\,dx}, \quad \text{as } u \to \infty \tag{3.1}$$

(ii) interlacing

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots \tag{3.2}$$

(iii) global condition

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^2 \prod_{k\neq n}^{\infty} \left(1-\frac{\mu_n^2}{\mu_k^2}\right) \prod_{k}^{\infty} \left(1-\frac{\mu_n^2}{\lambda_k^2}\right)} < \infty . \tag{3.3}$$

The asymptotic condition is easy to understand and can be deduced in a straightforward manner form the direct problem. The interlacing condition is also sensible: after all the μ's are associated with a configuration freer than that for the λ's. Less obvious is the fact that this condition insures the positivity of $\rho(x)$. This is connected to a theorem of Stieltjes (1894) which we shall discuss in the sequel. The third condition insures that the total mass of the string is finite. It plays an essential role in the construction of the solution to the inverse problem for a vibrating string. We would like to find the conditions for the beam analogous to (3.1) - (3.3) for the string.

Interlacing of eigenvalues. One would expect that the following ordering would hold:

$$\omega_1 < \sigma_1 < \nu_1 < \mu_1 < \lambda_1 < \ldots < \omega_i < \sigma_i < \nu_i < \mu_i < \lambda_i < \ldots$$

However, this is *not* the case. Indeed, by considering the homogeneous beam (i.e. ρ and r constants), we can see that the spectrum associated with the clamped/clamped configuration, namely $\{\lambda_n\}_1^{\infty}$, does not interlace with that for the free/clamped one, i.e. $\{\omega_n\}_1^{\infty}$ (Platzman, private communication).

To elucidate this question, we appeal to a paper of Krein (1939). If we were to apply the theorem stated in that paper to the case of the beam at hand, we would get:

$$\omega_i < \lambda_i < \omega_{i+2} , \tag{3.4a}$$

$$\sigma_i < \lambda_i < \sigma_{i+2} , \tag{3.4b}$$

$$\nu_i < \lambda_i < \nu_{i+2} , \tag{3.4c}$$

$$\mu_i < \lambda_i < \mu_{i+2} . \tag{3.4d}$$

These results are general results stemming from the properties of Green's functions of fourth order operators which are so-called "oscillating". These results can be improved for the particular case of the beam operator. The first step consists in examining the first eigenvalues of our five eigenvalue problems. We start by considering μ_1 and ν_1. To that effect we introduce a hybrid eigenvalue problem as follows:

$$\left.\begin{aligned} (ry'')'' &= \Lambda^2 \rho y , && 0 < x < L , \\ \alpha y-(1-\alpha)y' &= y'' = 0, && \text{at } x = 0 , \\ y = y' &= 0, && \text{at } x = L . \end{aligned}\right\} \tag{3.5}$$

where α is a parameter ranging over (0,1).

Clearly

$$\Lambda_n(0) = \nu_n , \tag{3.6a}$$

and

$$\Lambda_n(1) = \mu_n \quad . \tag{3.6b}$$

The adjoint problem, which will also enter into our discussion, is

$$\left.\begin{array}{ll} (r\eta'')'' = \Lambda^2 \rho \eta , & 0 < x < L , \\ \eta = \alpha(r\eta'') - (1-\alpha)(r\eta'')' = 0 , & \text{at } x = 0 , \\ \eta = \eta' = 0 , & \text{at } x = L . \end{array}\right\} \tag{3.7}$$

After differentiation with respect to the parameter α, we can easily see that

$$\frac{d\Lambda^2}{d\alpha} = - \frac{y'(0,\alpha)(r\eta'')(0,\alpha)}{\alpha(1-\alpha)\int_0^L \rho y \eta dx} \quad . \tag{3.8}$$

We shall specialize this result to the first eigensolutions of (3.5) and (3.7) and appeal to a result of Gantmakher* (1936), namely that

$$\left.\begin{array}{l} y_1(x,\alpha) > 0 , \\ \eta_1(x,\alpha) > 0 , \end{array}\right\} \quad \text{for } 0 < x < L \quad . \tag{3.9}$$

---

*Strictly speaking, we should prove first that (3.5) and (3.7) can be transformed into integral equations with oscillating kernels. This is indeed the case thanks to a theorem of Karlin (1971).

Consequently

$$y_1\ (0,\alpha) \geq 0$$

and in view of the boundary condition at x=0 ,

$$y_1'\ (0,\alpha) \geq 0\ . \tag{3.10}$$

We prove next that

$$(r\eta_1'')(0,\alpha) \leq 0\ . \tag{3.11}$$

Let us assume the contrary. Then, by continuity there exists an interval, say, $(0,\xi)$, over which

$$r\eta_1'' > 0\ .$$

This interval must be smaller than the length of the beam. Indeed, otherwise η_1' would be an increasing function throughout (0,L) and since $\eta_1'(0,\alpha)$ must be positive, it would follow that $\eta_1'(L,\alpha)$ is also positive. But this is impossible on account of the boundary condition. Thus $0 < \xi < L$. We can be more specific and define ξ such that

$$(r\eta_1'')(\xi,\alpha) = 0\ .$$

Now, integrating (3.7) twice from 0 to ξ, we see that

$$-[(r\eta_1'')(0,\alpha) + \xi(r\eta_1'')'(0,\alpha)] = \Lambda_1^2 \int_0^\xi (\xi-t)\rho\eta_1 dt\ ,$$

and since $(r\eta_1'')(0,\alpha)$ is positive, it follows from the boundary condition that $(r\eta_1'')'(0,\alpha)$ is also positive. Hence, the left hand side of the above equation is negative whereas the right hand side is positive. Therefore (3.11) must be true and as a result, the derivative of Λ_1^2 with respect to α is positive, i.e.

$$\nu_1 < \mu_1 . \tag{3.12}$$

By considering a problem similar to (3.5) but with the boundary conditions

$$\alpha y - (1-\alpha)y' = (ry'')' = 0 \qquad \text{at } x = 0$$

we could show that

$$\sigma_1 < \nu_1 . \tag{3.13}$$

In a similar vein, we could also prove that

$$\omega_1 < \sigma_1 . \tag{3.14}$$

Combining all of these results, we get

$$\omega_1 < \sigma_1 < \nu_1 < \mu_1 < \lambda_1 . \tag{3.15}$$

The technique used to establish (3.15) is not suitable for the higher eigenvalues. To order them I have found that the introduction of certain auxiliary functions is very helpful.

Auxiliary functions. Recall that $u(x,-s)$ and $v(x,-s)$ were fundamental solutions of the beam equation chosen in such a way as to satisfy the clamped boundary conditions at $x=L$ (see (1.7)). Recall also the forced problem (2.3)-(2.4) which led to the impulse response when $\rho(x)\eta(x)$ was set equal to $\delta(x)$. We can rewrite this problem as follows:

$$\left.\begin{aligned} (ry'')'' &= -s\rho y \ , \\ ry'' = (ry'')' - 1 &= 0 \ , \qquad \text{at } x=0 \ , \\ y = y' &= 0, \qquad \text{at } x=L \ . \end{aligned}\right\} \tag{3.16}$$

We can avail ourselves of the functions u and v to solve this problem. In fact

$$y(x,-s) = -\frac{u(x,-s)(rv'')(0,-s)-v(x,-s)(ru'')(0,-s)}{(ru'')(0,-s)(rv'')'(0,-s)-(rv')(0,-s)(ru'')'(0,-s)} \ , \tag{3.17}$$

and as a result, the impulse response given in (2.9) can also be written as

$$y(0,-s) = -\frac{u(0,-s)(rv'')(0,-s)-v(0,-s)(ru'')(0,-s)}{(ru'')(0,-s)(rv'')'(0,-s)-(rv'')(0,-s)(ru'')'(0,-s)} \tag{3.18a}$$

and

$$\theta(0,-s) = -\frac{u'(0,-s)(rv'')(0,-s)-v'(0,-s)(ru'')(0,-s)}{(ru'')(0,-s)(rv'')'(0,-s)-(rv'')(0,-s)(ru'')'(0,-s)} \ . \tag{3.18b}$$

These expressions suggest that we define the following functions:

$$Y(x,\omega^2) = u(x,\omega^2)(rv'')(x,\omega^2) - v(x,\omega^2)(ru'')(x,\omega^2) \ , \tag{3.19}$$

$$\Theta(x,\omega^2) = u'(x,\omega^2)(rv'')(x,\omega^2) - v'(x,\omega^2)(ru'')(x,\omega^2) \ , \tag{3.20}$$

$$D(x,\omega^2) = (ru'')(x,\omega^2)(rv'')'(x,\omega^2) - (rv'')(x,\omega^2)(ru'')'(x,\omega^2), \tag{3.21}$$

and that we work *directly* with them rather than with u and v. Numerically, this approach has the advantage of minimizing the loss of accuracy due to cancellations in the computation of Y, Θ and D.

We can look upon Y, Θ and D as determinants obtained by considering the 2 x 2 submatrices arising from the 4 x 2 matrix

$$\begin{bmatrix} u & v \\ u' & v' \\ ru'' & rv'' \\ (ru'')' & (rv'')' \end{bmatrix}.$$

Three other such determinants can be formed, namely

$$I(x,\omega^2) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} , \tag{3.22}$$

$$J(x,\omega^2) = \begin{vmatrix} u & v \\ (ru'')' & (rv'')' \end{vmatrix} , \tag{3.23}$$

and

$$K(x,\omega^2) = \begin{vmatrix} u' & v' \\ (ru'')' & (rv'')' \end{vmatrix} , \tag{3.24}$$

It is a simple matter to check that these six auxiliary functions satisfy the following differential equations:

$$I = \frac{1}{r} Y \ , \tag{3.25a}$$

$$Y' = \Theta + J \ , \tag{3.25b}$$

$$J' = K \ , \tag{3.25c}$$

$$\Theta' = K \ , \tag{3.25d}$$

$$K' = \frac{1}{r} D - \omega^2 \rho I \ , \tag{3.25e}$$

$$D' = -\omega^2 \rho Y \ , \tag{3.25f}$$

where primes indicate differentiation with respect to x. These auxiliary equations, rather than the beam equation, will occupy the center stage during the reconstruction procedure. Note that in view of (1.7), the boundary conditions associated with (3.25) are

$$I = Y = J = \Theta = K = D - r^2(L) = 0 \quad \text{at } x=L. \tag{3.26}$$

Keeping in mind the definitions (3.19)-(3.24) of the auxiliary functions, it is a simple matter to show that

$$I(0,\omega^2) = r^2(L)(F_o F_2 - F_1^{\,2}) \prod_{n=1}^{\infty} (1 - \frac{\omega^2}{\lambda_n^2}) \ , \tag{3.27a}$$

$$Y(0,\omega^2) = -r^2(L)F_2 \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\mu_n^2}\right) , \tag{3.27b}$$

$$J(0,\omega^2) = \Theta(0,\omega^2) = r^2(L)F_1 \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\nu_n^2}\right) , \tag{3.27c}$$

$$K(0,\omega^2) = -r^2(L)F_o \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\sigma_n^2}\right) , \tag{3.27d}$$

$$D(0,\omega^2) = r^2(L) \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\omega_n^2}\right) . \tag{3.27e}$$

In the above expressions F_1 and F_2 are the first and second moments of the flaccidity defined in (1.27) and (1.29) while F_o is the zeroth moment, namely

$$F_o = \int_0^L \frac{dt}{r(t)} . \tag{3.28}$$

Note that (3.25c) and (3.25d) together with the boundary conditions (3.26) imply that

$$\Theta(x,\omega^2) = J(x,\omega^2) . \tag{3.29}$$

This result is easy to understand: it means that the eigenfrequencies of the truncated beam (x,L) are identical if the boundary conditions at the left end are $y = (ry'')' = 0$ or $y' = ry'' = 0$. As we already

know, these two problems are adjoint of one another.

In deriving the expression (3.27a) for $I(0,\omega^2)$ we made use of an identity for the auxiliary functions which will play a crucial role in the sequel, namely

$$-I(x,\omega^2)D(x,\omega^2) + Y(x,\omega^2)K(x,\omega^2) - \Theta^2(x,\omega^2) = 0 \ . \tag{3.30}$$

This identity can be established from the differential equations (3.25). However, its character is algebraic in nature and it is best obtained from the very definitions of the auxiliary functions.

The product representations (3.27) for x=0, can be viewed as special cases of product representations for arbitrary x's. Indeed, the auxiliary functions are entire functions of ω^2 of order 1/4. Consequently, we can write

$$I(x,\omega^2)=r^2(L)\left[\int_x^L \frac{dt}{r}\int_x^L \frac{(t-x)^2}{r}dt - \left(\int_x^L \frac{t-x}{r}dt\right)^2\right]\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\lambda_n^2(x)}\right), \tag{3.27'a}$$

$$Y(x,\omega^2) = -r^2(L)\int_x^L \frac{(t-x)^2}{r}dt\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\mu_n^2(x)}\right) \tag{3.27'b}$$

$$\Theta(x,\omega^2) = r^2(L)\int_x^L \frac{t-x}{r}dt\prod_{n=1}^{\infty}\left(1-\frac{\omega^2}{\nu_n^2(x)}\right) \tag{3.27'c}$$

$$K(x,\omega^2) = -r^2(L) \int_x^L \frac{dt}{r} \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\sigma_n^2(x)}\right) \qquad (3.27'd)$$

$$D(x,\omega^2) = r^2(L) \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2}{\omega_n^2(x)}\right) . \qquad (3.27'e)$$

$\lambda_n(x)$, $\mu_n(x)$, $\nu_n(x)$, $\sigma_n(x)$ and $\omega_n(x)$ are the nth eigenfrequencies of a truncated beam (x,L) clamped at the right end and satisfying respectively the clamped, supported, non self-adjoint, Rayleigh and free boundary conditions at the left end. These eigenfrequencies are real, positive and simple as a consequence of the theory of oscillating kernels.

These eigenfrequencies have an additional property,--namely they are increasing functions of x. This is easy to understand physically: as the original beam is stripped-off of the section (0,x), its inertia is decreased and as a result the natural frequencies are increased. We sketch a proof of this statement, or more specifically of the fact that $d\lambda_n^2(x)/dx > 0$. (A second proof will be given in the sequel). To that effect consider the truncated beam (α,L). Then

$$(ry_n'')'' = \lambda_n^2 \rho\, y_n \quad , \qquad \alpha < x < L ,$$

$$y_n(\alpha) = y_n'(\alpha) = 0 ,$$

$$y_n(L) = y_n'(L) = 0 .$$

Indicating a derivative with respect to α by a dot, we deduce that

$$(r\dot{y}_n'')'' = 2\lambda_n\dot{\lambda}_n\rho y_n + \lambda_n^2\rho\dot{y}_n \tag{3.31}$$

and from the boundary conditions:

$$\dot{y}_n + y_n' = \dot{y}_n' + y_n'' = 0 \quad \text{at } x = \alpha \ ,$$

$$\dot{y}_n = \dot{y}_n' = 0 \quad \text{at } x = L \ .$$

Multiplying (3.31) by y_n and integrating over (α, L) we get:

$$-r(\alpha)y_n''(\alpha,\alpha)\dot{y}_n'(\alpha,\alpha) = 2\lambda_n\dot{\lambda}_n\|y_n\|^2$$

which, after some simple manipulations, yields

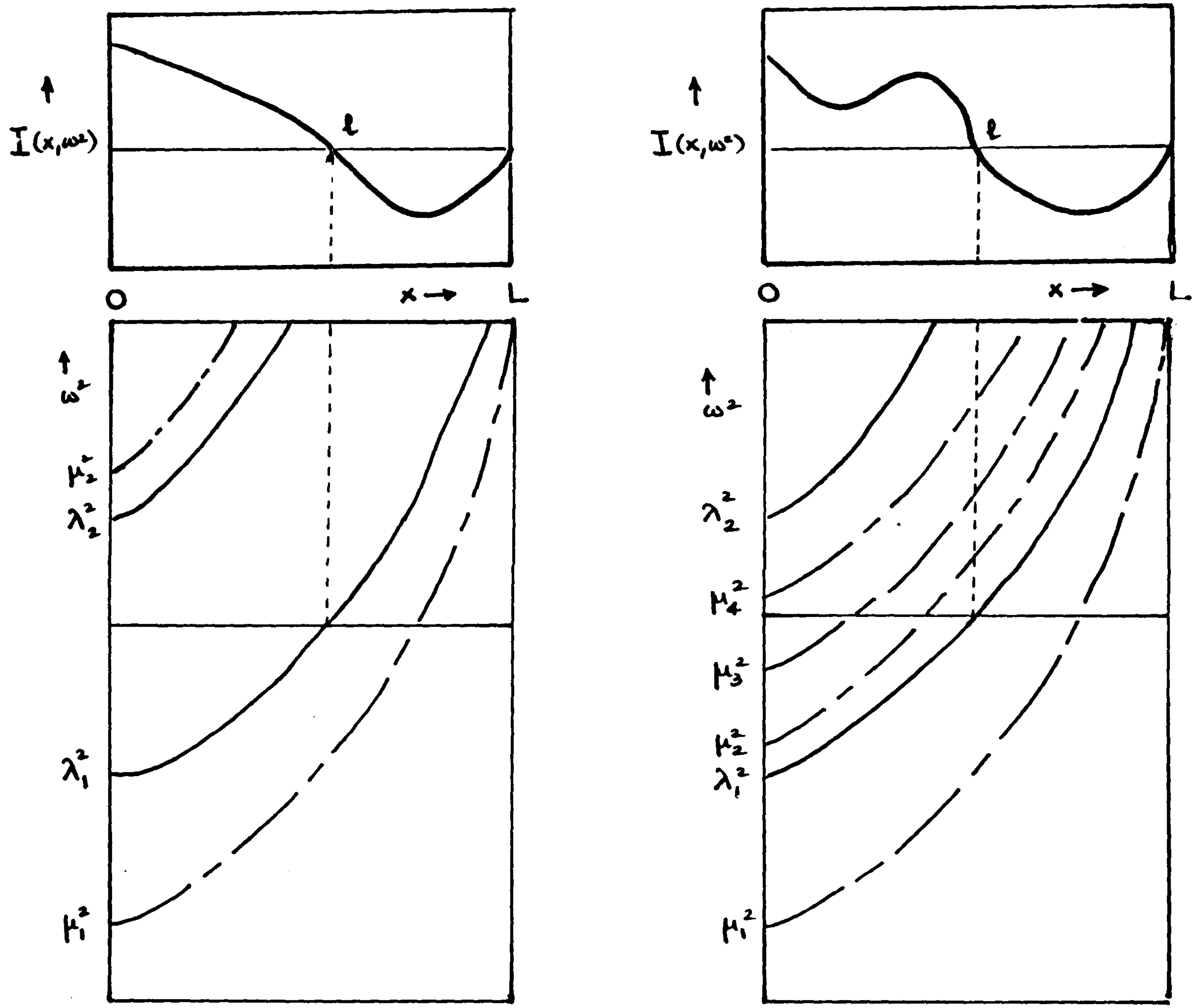
$$\frac{d\lambda_n(\alpha)}{d\alpha} = \frac{r(\alpha)[y_n''(\alpha,\alpha)]^2}{2\lambda_n(\alpha)\ \|y_n\|^2}$$

A similar procedure could be used for the other spectra.

The grand interlacing. We are now in a position to resume our discussion about the interlacing of the various spectra.

Let us consider the curve $I(x,\omega^2)$ for a fixed value of ω^2 lying between λ_1^2 and λ_2^2 (see Fig. 2).

Insert Fig. 2



I(x,ω²)
ℓ
O
x →
L
ω²
μ₂²
λ₂²
λ₁²
μ₁²
I(x,ω²)
ℓ
O
x →
L
ω²
λ₂²
μ₄²
μ₃²
μ₂²
λ₁²
μ₁²

Fig 2

$I(x,\omega^2)$ is positive for x in $(0,\ell)$ and negative for x in (ℓ,L) where

$$\lambda_1^2(\ell) = \omega^2.$$

(3.25a) implies that the zeros of I' coincide with the curves $\omega^2=\mu_n^2(x)$. Therefore I' vanishes only once in (ℓ,L) since an immediate generalization of (3.15) would yield

$$\mu_1(x) < \lambda_1(x) .$$

Over $(0,\ell)$, I' could either vanish or not vanish. If I' were different from zero for all x's in $(0,\ell(\omega^2))$ and for all ω's in the interval (λ_1,λ_2), then we would have

$$\mu_2 > \lambda_2$$

which contradicts (3.4d). On the other hand if I' vanished three or more times, then we would have

$$\mu_4 < \lambda_2$$

which also contradicts (3.4d). Therefore I' can vanish either once or twice in $(0,\ell)$. Let us examine the second case more closely. Graphically, the situation is as indicated in Fig. 3 . Consider the

Insert Fig. 3

the structure of I on the straight line $\omega = \lambda_2$. Certainly

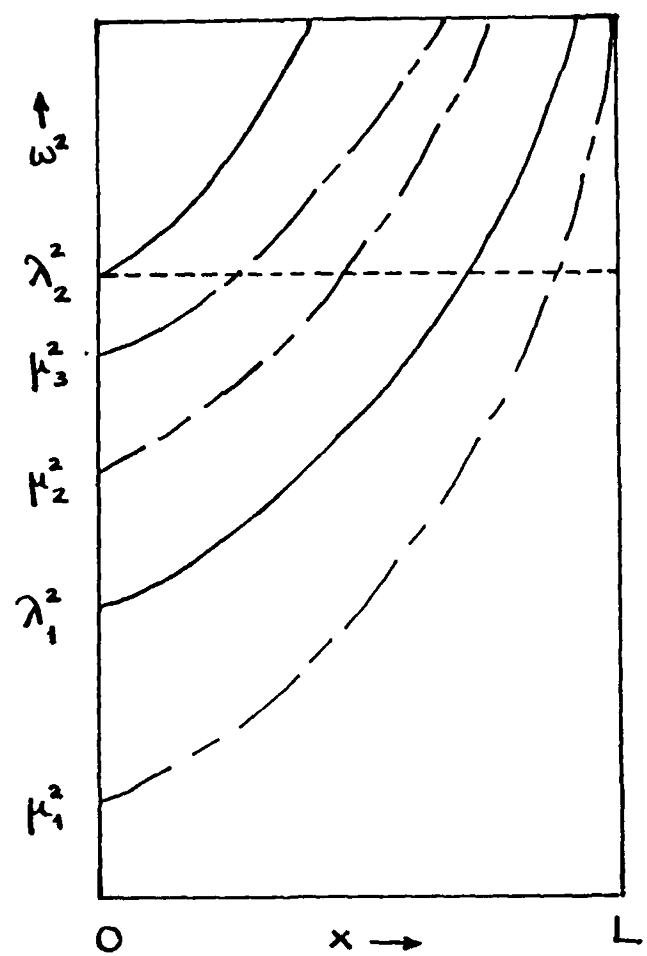


Fig. 3

$$I(0,\lambda_2^{\,2}) = 0 \;, \tag{3.32}$$

and

$$I(x,\lambda_2^{\,2}) \leq 0 \tag{3.33}$$

over the interval $(0,\ell(\lambda_2^{\,2}))$. Now

$$Y(0,\lambda_2^{\,2}) > 0 \;.$$

But, because of the auxiliary equation (3.25a), this inequality implies that

$$I'(0,\lambda_2^{\,2}) > 0$$

and hence for small values of x, it follows from (3.32) that

$$I(x,\lambda_2^{\,2}) > 0$$

which contradicts (3.33). Hence I' can only vanish once and as a result

$$\mu_1 < \lambda_1 < \mu_2 < \lambda_2$$

Repeating this procedure, we can conclude that

$$\mu_i < \lambda_i < \mu_{i+1} < \lambda_{i+1} \;. \tag{3.34}$$

By exploiting (3.25b), or rather

$$Y' = 2\Theta \ ,$$

we could show in a similar manner that

$$\nu_i < \mu_i < \nu_{i+1} < \mu_{i+1} \ . \tag{3.35}$$

Similarly, (3.25d) and (3.25f) would yield

$$\sigma_i < \nu_i < \sigma_{i+1} < \nu_{i+1} \tag{3.36}$$

and

$$\omega_i < \mu_i < \omega_{i+1} < \mu_{i+1} \ . \tag{3.37}$$

Additional interlacings can be obtained by means of the quadratic identity (3.30). Indeed, setting x=0 and $\omega=\omega_n$ in (3.30), we see that

$$Y(0,\omega_n^{\ 2})\ K(0,\omega_n^{\ 2}) - \Theta^2(0,\omega_n^{\ 2}) = 0 \ ,$$

which, together with (3.37) implies that

$$\omega_i < \sigma_i < \omega_{i+1} < \sigma_{i+1} \ . \tag{3.38}$$

Had we set x=0 and $\omega=\sigma_n$ in (3.30), we would have obtained

$$-I(0,\sigma_n^{\ 2})\ D(0,\sigma_n^{\ 2}) - \Theta^2(0,\sigma_n^{\ 2}) = 0 \ ,$$

which, together with (3.38) implies that

$$\sigma_i < \lambda_i < \sigma_{i+1} < \lambda_{i+1} . \tag{3.39}$$

By combining all of these interlacings we can write the *grand* interlacing

$$\dots < \sigma_i < \nu_i < \mu_i < \left({\lambda_i \atop \omega_{i+1}} \right) < \sigma_{i+1} < \nu_{i+1} < \mu_{i+1} < \left({\lambda_{i+1} \atop \omega_{i+1}} \right) < \dots \tag{3.40}$$

The bracket $\left({\lambda_i \atop \omega_{i+1}} \right)$ indicates that the order between these two eigenvalues cannot be decided. As we alluled to, the homogeneous beam provides an instance where one can check that the λ's and the ω's do not interlace.

Clearly, the data ought to satisfy (3.40) in order for a solution to exist. More specifically, since only $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$ are given, the data should satisfy the *small* interlacing, namely

$$\omega_1 < \nu_1 < \mu_1 < \omega_2 < \nu_2 < \mu_2 < \dots \tag{3.41}$$

We are prepared, of course, to require that the ω, ν and μ spectra have the asymptotic behavior given in (1.38) and perhaps some other gross condition(s) akin to (3.3) for the vibrating string. But, as we shall soon see, the situation is far more complicated and other conditions must come into play. Some of these additional conditions arise quite naturally in the process of constructing the solution to the inverse problem. Therefore, we must examine this process; the first step is a discretization of the direct problem.

A useful discretization. The basic idea for solving the inverse problem, is to look upon the beam with given spectra $\{\omega_n, \nu_n, \mu_n\}_1^\infty$ as the limit of a simpler beam with truncated spectra $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$ as the number of eigenfrequencies tends to infinity.

Beams with a finite number of eigenfrequencies must have a finite number of degrees of freedom. Thus, they must be made up by a finite number of point masses. Consequently, their density structure is of the form

$$\rho(x) = \sum_{i=1}^{N-1} m_i \, \delta(x-x_i). \tag{3.42}$$

The structure of the flexural rigidity is not dictated as clearly. One useful approximation is

$$\frac{1}{r(x)} = \sum_{i=1}^{N} f_i \, \delta(x-x_i) \; . \tag{3.43}$$

I have discussed already such discrete N- beams (Barcilon 1979a,b): they can be thought of as being made of N+1 weightless segments of length $\{\ell_i\}_o^N$ and of infinite rigidity, connected by clothespin-like devices of mass $\{m_i\}_1^{N-1}$ and "flaccidity" (or limpness) $\{f_i\}_1^N$ (see Fig. 4).

Insert Fig. 4

These elastic joints are located at $\{x_i\}_1^N$ where

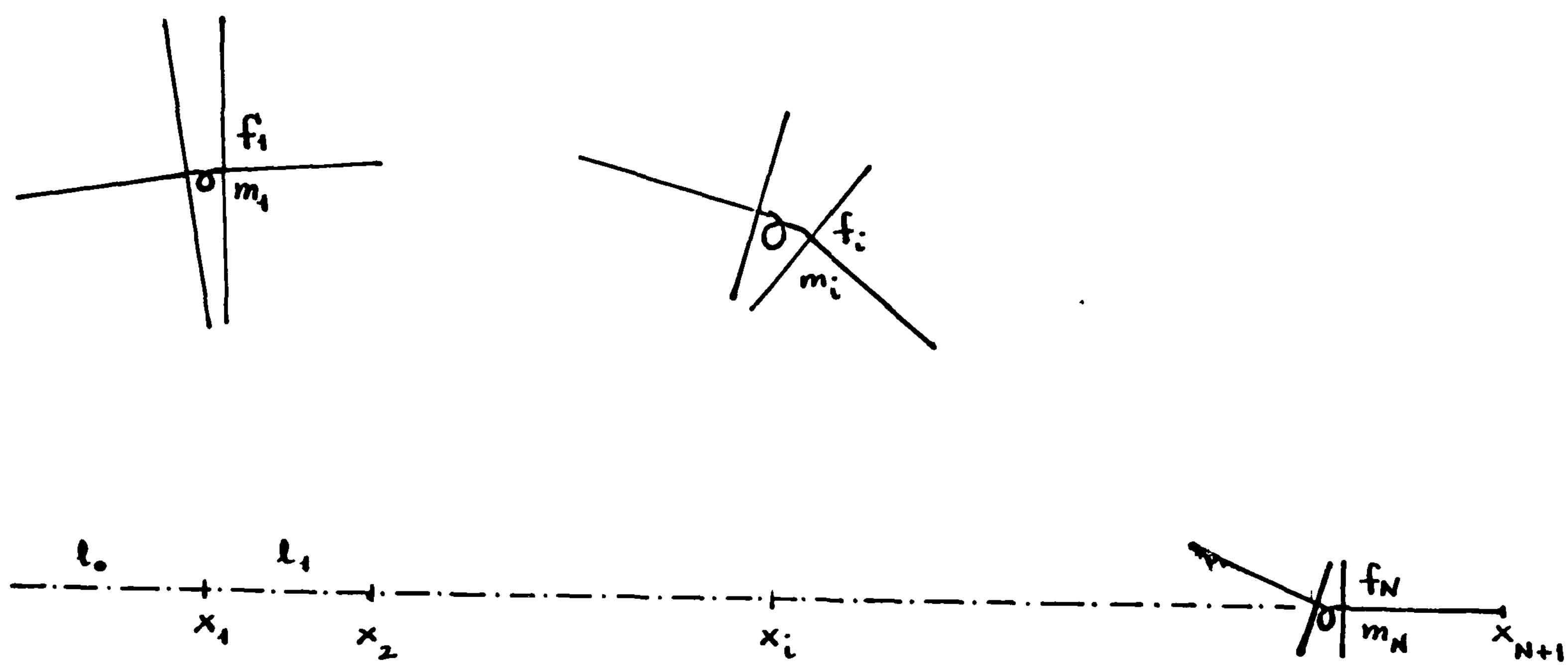


Fig. 4

$$x_{i+1} - x_i = \ell_i \ , \quad (i=0,\ 1,\ \ldots,\ N) \tag{3.44}$$

It is important to realize that the structure (3.42) of the density is solely responsible for the finite number of eigenfrequencies of the resulting beam. The structure (3.43) of the flaccidity is chosen for convenience. In fact, other forms could have been chosen which might indeed be preferable in other problems. For instance, we could have written

$$\frac{1}{r(x)} = \frac{1}{r_{i-1}} \quad \text{for} \quad x_{i-1} < x < x_i \ ,$$

i.e. we could have considered the weightless segments to have finite constant rigidity. This case is briefly discussed in an appendix.

Since we are interested in the discrete version of (3.25) it would seem expedient to substitute in these equations $\rho(x)$ and $r(x)$ by their expressions as given in (3.42) and (3.43). However, this approach is dangerous because of the occurrence of products of generalized functions. We shall follow therefore the long route and discretize (1.2) first.

Over a generic interval $(x_{i-1},\ x_i)$ the equations (1.2) become

$$\begin{aligned} y' &= \Theta \ , \\ \theta' &= 0 \ , \\ \tau' &= -\chi , \\ \chi' &= 0 \ . \end{aligned} \tag{3.45}$$

Integrating once, we see that for $x_{i-1} \leq \quad < x_i$,

$$
\begin{aligned}
y(x,\omega^2) &= \theta_{i-1}(\omega^2)\,[x-x_{i-1}] + y_{i-1}(\omega^2) \;, \\
\theta(x,\omega^2) &= \theta_{i-1}(\omega^2) \;, \\
\tau(x,\omega^2) &= -\chi_{i-1}(\omega^2)[x-x_{i-1}] + \tau_{i-1}(\omega^2) \;, \\
\chi(x,\omega^2) &= \chi_{i-1}(\omega^2) \;.
\end{aligned}
\tag{3.46}
$$

Clearly y_{i-1}, θ_{i-1}, τ_{i-1} and χ_{i-1} correspond to the values of y, θ, τ and χ at $x_{i-1}+0$. At $x=x_i$, the following jump conditions hold:

$$
\begin{aligned}
y(x_i+0,\omega^2) - y(x_i-0,\omega^2) &= 0 \;, \\
\theta(x_i+0,\omega^2) - \theta(x_i-0,\omega^2) &= f_i\tau_i \;, \\
\tau(x_i+0,\omega^2) - \tau(x_i-0,\omega^2) &= 0 \;, \\
\chi(x_i+0,\omega^2) - \chi(x_i-0,\omega^2) &= -m_i\omega^2 y_i \;.
\end{aligned}
\tag{3.47}
$$

Substituting the expressions (3.46) in the above jump conditions we obtain the discrete version of (1.2), namely

$$
\begin{aligned}
y_i &= y_{i-1} + \ell_{i-1}\,\theta_{i-1} \;, \\
\theta_i &= \theta_{i-1} + f_i\,\tau_i \;, \\
\tau_i &= \tau_{i-1} - \ell_{i-1}\,\chi_{i-1} \;, \\
\chi_i &= \chi_{i-1} - \omega^2 m_i\,y_i \;.
\end{aligned}
\tag{3.48}
$$

We next define two fundamental solutions of the above difference equations, clcsely related to the functions $u(x,\omega^2)$ and $v(x,\omega^2)$. We denote these solutions by a superscript (3) and (4), and characterize them by their values at $x=x_{N+1}$:

$$
\begin{aligned}
&y_{N+1}^{(3)} = \theta_{N+1}^{(3)} = \tau_{N+1}^{(3)} - 1 = \chi_{N+1}^{(3)} = 0 \quad , \\
&y_{N+1}^{(4)} = \theta_{N+1}^{(4)} = \tau_{N+1}^{(4)} = \chi_{N+1}^{(4)} + 1 = 0 \quad ,
\end{aligned}
\tag{3.49}
$$

or, in view of (3.46)

$$
\begin{aligned}
&y_{N}^{(3)} = \theta_{N}^{(3)} = \tau_{N}^{(3)} - 1 = \chi_{N}^{(3)} = 0 \; , \\
&y_{N}^{(4)} = \theta_{N}^{(4)} = \tau_{N}^{(4)} + \ell_N = \chi_{N+1}^{(4)} = 0 \; .
\end{aligned}
\tag{3.50}
$$

Pursuing the analogy with the continuous case, we define

$$
I_i(\omega^2) = \begin{vmatrix} y_i^{(3)} & y_i^{(4)} \\ \theta_i^{(3)} & \theta_i^{(4)} \end{vmatrix} \; , \tag{3.51a}
$$

$$
Y_i(\omega^2) = \begin{vmatrix} y_i^{(3)} & y_i^{(4)} \\ \tau_i^{(3)} & \tau_i^{(4)} \end{vmatrix} \; , \tag{3.51b}
$$

$$
J_i(\omega^2) = \begin{vmatrix} y_i^{(3)} & y_i^{(4)} \\ \chi_i^{(3)} & \chi_i^{(4)} \end{vmatrix} \; , \tag{3.51c}
$$

$$\Theta_i(\omega^2) = \begin{vmatrix} \Theta_i^{(3)} & \Theta_i^{(4)} \\ \tau_i^{(3)} & \tau_i^{(4)} \end{vmatrix} , \tag{3.51d}$$

$$K_i(\omega^2) = - \begin{vmatrix} \theta_i^{(3)} & \theta_i^{(4)} \\ \chi_i^{(3)} & \chi_i^{(4)} \end{vmatrix} , \tag{3.51e}$$

$$D_i(\omega^2) = - \begin{vmatrix} \tau_i^{(3)} & \tau_i^{(4)} \\ \chi_i^{(3)} & \chi_i^{(4)} \end{vmatrix} . \tag{3.51f}$$

Making use of (3.48), we can deduce that these discrete auxiliary variables satisfy the following difference equations:

$$I_i = I_{i-1} + f_i Y_i \quad , \tag{3.52a}$$

$$Y_i = Y_{i-1} + \ell_{i-1}\, \Theta_{i-1} + \ell_{i-1}\, J_i \quad ,$$

$$J_i = J_{i-1} + \ell_{i-1}\, K_{i-1} \quad , \tag{3.52c}$$

$$\Theta_i = \Theta_{i-1} + \ell_{i-1}\, K_{i-1} \quad , \tag{3.52d}$$

$$K_i = K_{i-1} + f_i D_i - \omega^2 m_i\, I_{i-1} \quad , \tag{3.52e}$$

$$D_i = D_{i-1} - \omega^2 m_i\, Y_i \quad . \tag{3.52f}$$

The advantage of following this procedure is now clear: we do not have to worry about whether the derivatives are to be discretized as forward, backward or centered differences. The choice is made quite naturally. As an additional bonus, we can check that the discrete version of the quadratic identity (3.30), namely

$$-I_i D_i + Y_i K_i - \Theta_i^{\,2} = 0 \tag{3.53}$$

is compatible with the equations (3.52). Equation (3.53) takes into account the face that

$$J_i = \Theta_i \quad , \tag{3.54}$$

which follows from (3.52c), (3.52d) and the end conditions

$$I_N = Y_N = J_N = \Theta_N = K_N = D_N - 1 = 0 \ . \tag{3.55}$$

Eventhough the generic N-beam under discussion has a length x_{N+1}, we ended up applying the boundary conditions at x_N. As a matter of fact, the length of the last infinitely rigid, weightless segment ℓ_N as well as the mass of the clast clothespin m_N do not enter into the equations of the discrete beam. This is a consequence of the clamped boundary conditions at the right end. Indeed, the last segment remains stationary during the vibration. Similarly, the last clothespin flexes but does not move up or down. This situation is reminiscent of that for the pathological beams previously discussed.

Starting from the values of the discrete auxiliary variables at $x = x_N$ and making use of (3.52), we can show that $I_i(\omega^2)$ is a polynomial in ω^2 of degree N-i-2 whereas $Y_i(\omega^2)$, $\Theta_i(\omega^2)$, $K_i(\omega^2)$ and $D_i(\omega^2)$ are polynomials of degree N-i-1. In particular, at the point x=0 which is associated with i=0, we have

$$I_o(\omega^2) = \left[\sum_1^N f_i \sum_1^N x_i^2 f_i - \left(\sum_1^N x_i f_i \right)^2 \right] \prod_{n=1}^{N-2} \left(1 - \frac{\omega}{\lambda_n^2}\right) , \tag{3.56a}$$

$$Y_o(\omega^2) = -\left[\sum_1^N x_i^2 f_i \right] \prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\mu_n^2}\right) , \tag{3.56b}$$

$$\Theta_o(\omega^2) = \left[\sum_1^N x_i f_i \right] \prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\nu_n^2}\right) , \tag{3.56c}$$

$$K_o(\omega^2) = -\left\{ \sum_1^N f_i \right\} \prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\sigma_n^2}\right) , \tag{3.56d}$$

$$D_o(\omega^2) = \prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\omega_n^2}\right) . \tag{3.56e}$$

These expressions are the discrete analogues of (3.27). We are now in a position to outline a program for solving the inverse problem. Given the impulse response, i.e. given $\{\omega_n, \nu_n, \mu_n\}_1^\infty$, F_1 and F_2, we shall attempt to construct that N-beam whose eigenfrequencies are $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$ and such that

$$\sum_1^N x_i f_i = F_1 \quad ,$$

$$\sum_1^N x_i^2 f_i = F_2 \quad . \tag{3.57}$$

In other words, given the 3N-1 bits of data $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$, F_1, F_2 we set out to find the 3N-1 unknowns $\{x_i\}_1^N$, $\{m_i\}_1^{N-1}$, $\{f_i\}_1^N$.

<u>Construction of $I_o(\omega^2)$ and $K_o(\omega^2)$.</u> With the given data, we can construct the polynomials $Y_o(\omega^2)$, $\Theta_o(\omega^2)$ and $D_o(\omega^2)$. By means of the polynomials and the quadratic identity (3.53) for i=0, we can find the value of I_o at N-1 points. Indeed, by setting $\omega=\mu_n$ in (3.53) we deduce that

$$I_o(\mu_n^2) = - \frac{\Theta_o^2(\mu_n^2)}{D_o(\mu_n^2)} \; , \quad n=1, 2, \ldots, N-1 \tag{3.58}$$

$I_o(\omega^2)$, which as indicated by (3.56a) is a polynomial of degree N-2, is therefore completely determined:

$$I_o(\omega^2) = - \sum_{n=1}^{N-1} \frac{\Theta_o^2(\mu_n^2)}{D_o(\mu_n^2)} \frac{\prod_{k\neq n}^{N-1} (1-\frac{\omega^2}{\mu_k^2})}{\prod_{k\neq n}^{N-1} (1-\frac{\mu_n^2}{\mu_k^2})} \; . \tag{3.59}$$

Note that the zeros of the above polynomial are real and positive. Indeed, since ω_i and μ_i interlace, the terms in the sequence $\{D_o(\mu_n^2)\}_1^{N-1}$

alternate in sign. As a result the sign of $I_o(\mu_n^2)$ alternates with n. We shall denote the zeros of $I_o(\omega^2)$ by $\{\lambda_n^{(N)}\}_1^{N-2}$, the superscript N being used to remind us that these eigenvalues are associated with the N-beam. We shall drop this superscript whenever there is no possible source of confusion.

We have just outlined a procedure for finding the eigenfrequencies of the N-beam in the clamped/clamped configuration. The fact that we can find these eigenfrequencies is not surprising since F_1, F_2 and $\{\omega_n, \nu_n, \mu_n\}_1^{N-1}$ define the N-beam completely. What is surprising is that we can do it without having to first solve for the structure of the N-beam! In the same way, we can deduce $K_o(\omega^2)$ and its zeros. We should point out that the fact that we were successful in determining $I_o(\omega^2)$ and $K_o(\omega^2)$ is related to the way in which the given spectra entered into the quadratic identity. As discussed in Barcilon (1979b) three spectra and two gross constants a sufficient to determine a non-pathologic beam, provided that these spectra are *sympathetic*, i.e. provided that they are such as to yield two other spectra from the quadratic identity.

Having established the interlacing of $\{\mu_n\}_1^{N-1}$ and $\{\lambda_n^{(N)}\}_1^{N-2}$, i.e.

$$\mu_1 < \lambda_1^{(N)} < \dots \qquad < \lambda_{N-2}^{(N)} < \mu_{N-1} \quad , \tag{3.60}$$

we can establish in a similar manner that

$$\omega_1 < \sigma_1^{(N)} < \dots \qquad < \omega_{N-1} < \sigma_{N-1}^{(N)} \quad , \tag{3.61}$$

and

$$\sigma_1^{(N)} < \lambda_1^{(N)} < \ldots \qquad < \lambda_{N-2}^{(N)} < \sigma_{N-1}^{(N)} \quad . \tag{3.62}$$

To these interlacing relations, we can add those that the data must satisfy, i.e. the small interlacing (3.41), or rather

$$\omega_1 < \nu_1 < \mu_1 < \ldots \qquad < \omega_{N-1} < \nu_{N-1} < \mu_{N-1} \quad . \tag{3.63}$$

However, the grand interlacing (3.40) is equivalent to nine interlacing conditions; we are missing the following three:

$$\sigma_1^{(N)} < \nu_1 < \ldots \qquad < \sigma_{N-1}^{(N)} < \nu_{N-1} \quad , \tag{3.64a}$$

$$\sigma_1^{(N)} < \mu_1 < \ldots \qquad < \sigma_{N-1}^{(N)} < \mu_{N-1} \quad , \tag{3.64b}$$

and

$$\nu_1 < \lambda_1^{(N)} < \ldots \qquad < \lambda_{N-2}^{(N)} < \nu_{N-1} \quad . \tag{3.64c}$$

These three interlacing conditions are related to each other. In fact, (3.64c) implies (3.64a) and (3.64b). One can see this by writing the quadratic identity as follows:

$$I_o(\nu_n^{\,2})\; D_o(\nu_n^{\,2}) = Y_o(\nu_n^{\,2})\; K_o(\nu_n^{\,2}) \quad . \tag{3.65}$$

Then (3.64c), together with (3.63), implies that $K_o(\nu_n^{\,2})$ alternates in sign, i.e. (3.64a) holds. In addition, (3.64a) and (3.61) imply

(3.64b). Thus, we only need to focus our attention on (3.64c). This interlacing condition *cannot* be deduced from (3.63). We present in the appendix an example of a trio of spectra satisfying the small interlacing, which give rise to a λ-spectrum which violates (3.64c). Therefore, *not all interlacing sequences* $\{\omega_n\}_1^{N-1}$, $\{\nu_n\}_1^{N-1}$ and $\{\mu_n\}_1^{N-1}$ *are bona fide spectra of an N-beam.* (This is to be contrasted with the situation for an N-string). To be such, these sequences must satisfy the additional constraints

$$(-1)^n \, I_o(\nu_n^2) < 0 \ , \qquad n=1,\ 2,\ \ldots,\ N-1 \tag{3.66}$$

or, in terms of the data,

$$(-1)^n \sum_{j=1}^{N-1} \frac{\prod_{i=1}^{N-1} \left(1-\frac{\mu_j^2}{\nu_i^2}\right)^2 \prod_{i\neq j}^{N-1} \left(1-\frac{\nu_n^2}{\mu_i^2}\right)}{\prod_{i=1}^{N-1} \left(1-\frac{\mu_j^2}{\omega_i^2}\right) \prod_{i\neq j}^{N-1} \left(1-\frac{\mu_j^2}{\mu_i^2}\right)} < 0 \ .$$

<u>The stripping procedure</u>. Let us assume that we have determined $I_o(\omega^2)$ and $K_o(\omega^2)$, and that $\lambda_n^{(N)}$ and $\sigma_n^{(N)}$ thus found satisfy all the necessary interlacing conditions. The next step consists in writing (3.52) for i=1 as follows:

$$-\frac{\Theta_o(\omega^2)}{K_o(\omega^2)} = \ell_o - \frac{\Theta_1(\omega^2)}{K_o(\omega^2)} \ , \tag{3.67a}$$

$$Y_1(\omega^2) = Y_o(\omega^2) + \ell_o\Theta_o(\omega^2) + \ell_o\Theta_1(\omega^2) \quad , \tag{3.67b}$$

$$-\frac{I_o(\omega^2)}{Y_1(\omega^2)} = f_1 - \frac{I_1(\omega^2)}{Y_1(\omega^2)} \quad , \tag{3.67c}$$

$$\frac{D_o(\omega^2)}{Y_1(\omega^2)} = m_1\omega^2 + \frac{D_1(\omega^2)}{Y_1(\omega^2)} \quad , \tag{3.67d}$$

$$K_1(\omega^2) = K_o(\omega^2) + f_1D_1(\omega^2) - m_1\omega^2I_o(\omega^2) \quad . \tag{3.67e}$$

By dividing $-\Theta_o(\omega^2)$ by $K_o(\omega^2)$, which are two polynomials of the same degree, the quotient is a constant ℓ_o and remainder a polynomial of degree N-2 in ω^2, namely $\Theta_1(\omega^2)$. Knowing ℓ_o and $\Theta_1(\omega^2)$, we can find $Y_1(\omega^2)$. Dividing next $I_o(\omega^2)$ by $-Y_1(\omega^2)$, once again two polynomials of the same degree, we infer the quotient f_1 and the remainder $I_1(\omega^2)$. Similarly, by dividing $D_o(\omega^2)$ by $Y_1(\omega^2)$ which is of lower degree, the quotient is $m_1\omega^2$ and the remainder is $D_1(\omega^2)$. Finally, we can evaluate $K_1(\omega^2)$ and start the cycle over again.

We postpone a discussion of this formal construction procedure until later and simply remark at this stage that by setting $\omega=0$ in (3.67d), (3.67e), (3.67a) and (3.67b) we would get:

$$D_1(0) = 1 \quad ,$$

$$K_1(0) = -F_o^{(N)} + f_1 = -\sum_2^N f_i \quad ,$$

$$\Theta_1(0) = F_1 - \ell_o F_o^{(N)} = \sum_2^N (x_i - \ell_o)\, f_i \quad , \tag{3.68}$$

$$Y_1(0) = -F_2 + 2\ell_o F_1 - \ell_o^{\,2} F_o^{(N)} \quad ,$$

$$= -\sum_2^N (x_i - \ell_o)^2\, f_i \quad ,$$

where

$$F_o^{(N)} = \sum_1^N f_i \; .$$

$I_1(0)$ can be deduced from the quadratic identity:

$$I_1(0) = \sum_2^N f_i \sum_2^N (x_i - \ell_o)^2 f_i - \Big(\sum_2^N (x_i - \ell_o)\, f_i\Big)^2 \; .$$

These expressions are similar to those for $D_o(0)$, ..., $I_o(0)$ except for the fact that the first segment as well as the first clothespin are missing. Thus, by means of this procedure we have "stripped-off" a

little segment of the beam. This is reminiscent of other inverse problems such as in seismic prospecting where this stripping-off is usually carried out in the time domain (Berkhout & van Wulfften Palthe, 1979).

<u>The Stieltjes theorem</u>. The procedure we have outlined is formal since it does not necessarily yield values of ℓ_i, m_i and f_i which are positive and hence physically meaningful. Indeed, numerical experiments carried out with spectra satisfying the grand interlacing have revealed that conditions (3.41) and (3.66) are *not* sufficient to guarantee the positivity of the physical characteristics of the N-beam. In contrast, the interlacing (3.2) is sufficient to guarantee the positivity of the density of the corresponding vibrating string. This is a direct result of a theorem of Stieltjes which we can state thus:

Part 1. If $0 < \beta_1 < \alpha_1 < \dots < \beta_{N-1} < \alpha_{N-1}$, then

$$\frac{\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\alpha_n^2}\right)}{\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\beta_n^2}\right)} = \ell + \frac{\prod_{n=1}^{N-2}\left(1-\frac{\omega^2}{\gamma_n^2}\right)}{\prod_{n=1}^{N-1}\left(1-\frac{\omega^2}{\beta_n^2}\right)} , \tag{3.69}$$

where

$$\ell > 0$$

and

$$\alpha_i < \gamma_i < \beta_{i+1} , \qquad i=1, 2, \dots, N-2 . \tag{3.70}$$

Part 2. If $0 < \beta_1 < \gamma_1 < \dots < \gamma_{N-2} < \beta_{N-1}$, then

$$\frac{\prod_{n=1}^{N-1} \left(1 - \frac{\omega^2}{\beta_n^2}\right)}{\prod_{n=1}^{N-2} \left(1 - \frac{\omega^2}{\gamma_n^2}\right)} = -m\omega^2 + \frac{\prod_{n=1}^{N-2} \left(1 - \frac{\omega^2}{\delta_n^2}\right)}{\prod_{n=1}^{N-2} \left(1 - \frac{\omega^2}{\gamma_n^2}\right)}, \tag{3.71}$$

where

$$m > 0$$

and

$$\beta_i < \delta_i < \gamma_i, \qquad i=1,\ 2,\ \dots,\ N-2\ . \tag{3.72}$$

We can apply Part 1 of Stieltjes theorem to (3.67a) to show that $\ell_o > 0$. In fact

$$\ell_o = \frac{F_1}{F_o^{(N)}} \prod_{n=1}^{N-1} \frac{\sigma^{(N)^2}}{\nu_n^2}, \tag{3.73}$$

or, equivalently

$$\ell_o = \frac{F_2}{F_1} \prod_{n=1}^{N-1} \frac{\nu_n^2}{\mu_n^2}. \tag{3.74}$$

Both of the above expressions for ℓ_o are compatible on account of the quadratic identity. Also, if we denote the zeros of $\Theta_1(\omega^2)$ by $\{\nu_n'^2\}_1^{N-2}$, then (3.70) implies that

$$\nu_i < \nu_i' < \sigma_{i+1} , \qquad i=1, 2, \ldots, N-2 . \tag{3.75}$$

Eliminating ℓ_o from (3.67b), we get

$$Y_1(\omega^2) = Y_o(\omega^2) - \frac{\Theta_o^{\,2}(\omega^2)}{K_o(\omega^2)} + \frac{\Theta_1^{\,2}(\omega^2)}{K_o(\omega^2)} ,$$

which, because of the quadratic identity, we can also write as

$$Y_1(\omega^2) = \frac{I_o(\omega^2)D_o(\omega^2)}{K_o(\omega^2)} + \frac{\Theta_1^{\,2}(\omega^2)}{K_o(\omega^2)} .$$

By setting $\omega=\lambda_i$ in the above equation, we deduce that

$$Y_1(\lambda_i^{\,2}) = \frac{\Theta_1^{\,2}(\lambda_i^{\,2})}{K_o(\lambda_i^{\,2})} , \qquad i=1, 2, \ldots, N-1 .$$

Since σ_i and λ_i interlace, $K_o(\lambda_i^{\,2})$ changes sign as i goes from 1 to N-1. Consequently the zeros μ_i' of $Y_1(\omega^2)$ and λ_i interlace. In fact, we can show that

$$\mu_1' < \lambda_1 < \ldots < \mu_{N-2}' < \lambda_{N-2} . \tag{3.76}$$

This is the condition which is necessary if we were to apply Part 1 of Stieltjes theorem to (3.67c). As a consequence it follows that $f_1 > 0$ and that

$$\lambda_i < \lambda'_i < \mu'_{i+1} , \qquad i=1,\ 2,\ \ldots,\ N-3 . \tag{3.77}$$

In fact

$$f_i = \frac{F_o^{(N)} F_2 - F_1^{\ 2}}{F_2'} \prod_{n=1}^{N-2} \frac{\mu'^{\,2}_n}{\lambda_n^{\ 2}} , \tag{3.78}$$

where

$$F'_2 = \sum_2^N (x_i - \ell_o)^2 f_i .$$

A more useful expression for f_1 can be obtained by writing (3.52e) thus:

$$K_1 = K_o + f_1 D_o - \omega^2 m_1 I_1 ,$$

from which we deduce that

$$f_1 = F_o^{(N)} \prod_{n=1}^{N-1} \frac{\omega_n^{\ 2}}{\sigma_n^{(N)2}} \tag{3.79}$$

Incidentally, as a result of the interlacing between ω_n and $\sigma_n^{(N)}$, f_1 is smaller than $F_o^{(N)}$. Hence $K_1(0)$ as given by (3.68) is negative. In the same vein, (3.73) shows that $\ell_o F_o^{(N)}$ is smaller than F_1 and hence $\Theta_1(0)$ as given by (3.68) is positive. Finally, in this same formula, $Y_1(0)$ can be seen to be negative. In summary

$$K_1(0) < 0 \,,$$

$$\Theta_1(0) > 0 \,, \tag{3.80}$$

$$Y_1(0) < 0 \,,$$

i.e., they have the same signs as $K_o(0)$, $\Theta_o(0)$ and $Y_o(0)$ respectively. We next turn to the equation (3.67c) for K_1 and proceed to eliminate f_1 and m_1 from it. In other words, we write it thus:

$$K_1 = K_o + \frac{I_1 - I_o}{Y_1} D_1 - \frac{D_o - D_1}{Y_1} I_o$$

which after simplifications brought about by the quadratic identity for i=1, becomes

$$- D_o I_o + Y_1 K_o - \Theta_1^2 = 0 \,. \tag{3.81}$$

Setting $\omega = \mu'_i$ in this hybrid quadratic identity, we see that

$$D_o(\mu_i'^2) = - \frac{\Theta_1^2(\mu_i'^2)}{I_o(\mu_i'^2)} \,.$$

Making use of (3.76) we deduce that

$$\omega'_i < \mu'_i < \omega_{i+1} , \qquad i=1, 2, \ldots, N-2 . \tag{3.82}$$

We are now in a position to apply Part 2 of Stieltjes theorem to (3.67d). As a result $m_1 > 0$ and

$$\omega_i < \omega'_i < \mu'_1 , \qquad i=1, 2, \ldots, N-2 . \tag{3.83}$$

In fact

$$m_1 = \frac{1}{F'_2} \frac{\prod_{n=1}^{N-2} \mu_n'^2}{\prod_{n=1}^{N-1} \omega_n^2} . \tag{3.84}$$

For the record, we also write an alternative formula for m_1, namely

$$m_1 = \frac{F_o^{(N)}}{F_o^{(N)} F_2 - F_1^2} \prod_{n=1}^{N-1} \frac{\omega_n^2}{\sigma_n^{(N)2}} \tag{3.85}$$

We can derive two more interlacings by setting ω equal to ω_i' and λ_i' in the quadratic identity. These are

$$\omega_i' < \sigma_i' < \omega'_{i+1} , \qquad i=1, 2, \ldots, N-2 \tag{3.86}$$

and

$$\sigma_i' < \lambda_i' < \sigma_{i+1}' \quad , \qquad i=1,\ 2,\ \ldots,\ N-2 \quad . \tag{3.87}$$

In summary, starting with the polynomials $I_o(\omega^2)$, ..., $D_o(\omega^2)$ whose zeros satisfy the grand interlacing (3.40) we can deduce

(i) ℓ_o, f_1 and m_1 which are positive,

(ii) new polynomials $I_1(\omega^2)$, ..., $D_1(\omega^2)$ of lower degree; these polynomials have the same structure as the original ones at $\omega=0$. However, their zeros satisfy only *four* interlacing conditions. These conditions are

$$\left.\begin{array}{ll} \mu_1' < \lambda_1' < \ldots & < \lambda_{N-3}' < \mu_{N-2}' \ , \\ \sigma_1' < \lambda_1' < \ldots & < \lambda_{N-3}' < \sigma_{N-2}' \ , \\ \omega_1' < \mu_1' < \ldots & < \omega_{N-2}' < \mu_{N-2}' \ , \\ \omega_1' < \sigma_1' < \ldots & < \omega_{N-2}' < \sigma_{N-2}' \ . \end{array}\right\} \tag{3.88}$$

Thus, we do not close the circle: other conditions must be placed on the data to insure that the eigenvalues of the "stripped" beam satisfy the grand interlacing.

Another method for reconstructing the N-beam. We return to the continuous beam and to equations (3.27'). After stripping-off the portion $(0,x)$ from the original beam, its nth eigenfrequency in the clamped/clamped configuration will be $\lambda_n(x)$ such that

$$I(x,\lambda_n^2(x)) = 0 \ . \tag{3.89}$$

Varying x a little, we see that

$$I'(x,\lambda_n^2(x)) + \dot{I}(x,\lambda_n^2(x)) \frac{d\lambda_n^2}{dx} = 0 \quad , \tag{3.90}$$

where a dot represents a differentiation with respect to ω^2. But according to (3.27'a)

$$\dot{I}(x,\lambda_n^2(x) = - \frac{r^2(L)}{\lambda_n^2(x)} \left[\int_x^L \frac{dt}{r} \int_x^L \frac{(t-x)^2}{r} dt - \left(\int_x^L \frac{t-x}{r} dt \right)^2 \right] .$$

$$\cdot \prod_{k \neq n} \left(1 - \frac{\lambda_n^2(x)}{\lambda_k^2(x)}\right) \quad , \tag{3.91}$$

and I' can be obtained from the auxiliary equations (3.25).

Consequently

$$\frac{d\lambda_n^2}{dx} = \frac{\lambda_n^2(x)}{r(x)} \cdot \frac{\int_x^L \frac{(t-x)^2}{r}\,dt}{\int_x^L \frac{dt}{r}\int_x^L \frac{(t-x)^2}{r}\,dt - \left(\int_x^L \frac{t-x}{r}\,dt\right)^2} \cdot \prod_{k\neq n} \frac{1-\frac{\lambda_n^2(x)}{\mu_k^2(x)}}{1-\frac{\lambda_n^2(x)}{\lambda_k^2(x)}} \cdot$$

$$\cdot \left(\frac{\lambda_n^2(x)}{\mu_n^2(x)} - 1\right) \qquad (3.92)$$

Note that each term in the right hand side is positive. Hence $\lambda_n(x)$ is an increasing function of x. The reader may recall that we gave a different proof of this result earlier.

In the same manner, we can show that

$$\frac{d\mu_n^2}{dx} = 2\mu_n^2(x) \cdot \frac{\int_x^L \frac{t-x}{r}\,dt}{\int_x^L \frac{(t-x)^2}{r}\,dt} \cdot \prod_{k\neq n} \frac{1-\frac{\mu_n^2(x)}{\nu_k^2(x)}}{1-\frac{\mu_n^2(x)}{\mu_k^2(x)}} \cdot \left(\frac{\mu_n^2(x)}{\nu_n^2(x)} - 1\right), \qquad (3.93)$$

$$\frac{d\nu_n^2}{dx} = \nu_n^2(x) \int_x^L \frac{dt}{r} \cdot \prod_{k\neq n} \frac{1-\frac{\nu_n^2(x)}{\sigma_k^2(x)}}{1-\frac{\nu_n^2(x)}{\nu_k^2(x)}} \cdot \left(\frac{\nu_n^2(x)}{\sigma_n^2(x)} - 1\right), \qquad (3.94)$$

$$\frac{d\sigma_n^2}{dx} = \frac{\sigma_n^2(x)}{r(x)} \cdot \frac{1}{\int_x^L \frac{dt}{r}} \cdot \prod_{k\neq n} \frac{1-\frac{\sigma_n^2(x)}{\omega_k^2(x)}}{1-\frac{\sigma_n^2(x)}{\sigma_k^2(x)}} \cdot \left(\frac{\sigma_n^2(x)}{\omega_n^2(x)} - 1\right)$$

$$+ \frac{\sigma_n^4(x)\rho(x)}{\int_x^L \frac{dt}{r}} \cdot \left[\int_x^L \frac{dt}{r} \int_x^L \frac{(t-x)^2}{r} dt - \left(\int_x^L \frac{t-x}{r} dt\right)^2 \right] \prod_{k\neq n} \frac{1-\frac{\sigma_n^2(x)}{\lambda_k^2(x)}}{1-\frac{\sigma_n^2(x)}{\sigma_k^2(x)}} \cdot \left(1- \frac{\sigma_n^2(x)}{\lambda_n^2(x)}\right), \tag{3.95}$$

$$\frac{d\omega_n^2}{dx} = \rho(x)\omega_n^2(x) \cdot \int_x^L \frac{(t-x)^2}{r} dt \cdot \prod_{k\neq n} \frac{1-\frac{\omega_n^2(x)}{\mu_k^2(x)}}{1-\frac{\omega_n^2(x)}{\omega_k^2(x)}} \cdot \left(1- \frac{\omega_n^2(x)}{\mu_n^2(x)}\right) . \tag{3.96}$$

Our previous discretization for the N-beam has taught us (i) that m_N and ℓ_N cannot be retrieved and (ii) that there are N-2 eigenvalues for the clamped/clamped configuration as opposed to N-1 for the others. Keeping in mind these lessons, let us discretize the above formulas thus:

$$\lambda'^{2}_{n} = \lambda_n^2 - f_1\lambda_n^2 \cdot \frac{F_2}{F_oF_2-F_1^2} \cdot \frac{\prod_{k=1}^{N-1}\left(1-\frac{\lambda_k^2}{\mu_k^2}\right)}{\prod_{k\neq n}^{N-2}\left(1-\frac{\lambda_n^2}{\lambda_k^2}\right)} , \tag{3.92'}$$

$$\mu'^{2}_{n} = \mu_n^2 - 2\ell_o\mu_n \frac{F_1}{F_2} \cdot \frac{\prod_{k=1}^{N-1}\left(1-\frac{\mu_n^2}{\nu_k^2}\right)}{\prod_{k\neq n}^{N-1}\left(1-\frac{\mu_n^2}{\mu_k^2}\right)} , \tag{3.93'}$$

$$\nu'^{2}_{n} = \nu_n^2 - \ell_o\nu_n^2 F_o \cdot \frac{\prod_{k=1}^{N-1}\left(1-\frac{\nu_n^2}{\sigma_k^2}\right)}{\prod_{k\neq n}^{N-1}\left(1-\frac{\nu_n^2}{\nu_k^2}\right)} , \tag{3.94'}$$

$$\sigma'^{2}_{n} = \sigma_n^2 - f_1\sigma_n^2 \frac{1}{F_o} \cdot \frac{\prod_{k=1}^{N-1}\left(1-\frac{\sigma_n^2}{\omega_k^2}\right)}{\prod_{k\neq n}^{N-1}\left(1-\frac{\sigma_n^2}{\sigma_k^2}\right)}$$

$$+ m_1\sigma_n^4 \cdot \frac{F_oF_2-F_1^2}{F_o} \cdot \frac{\prod_{k=1}^{N-2}\left(1-\frac{\sigma_n^2}{\lambda_k^2}\right)}{\prod_{k\neq n}^{N-1}\left(1-\frac{\sigma_n^2}{\sigma_k^2}\right)} , \tag{3.95'}$$

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$$\omega'_n{}^2 = \omega_n^2 + m_1\omega_n^2 F_2 \cdot \frac{\prod_{k=1}^{N-1} \left(1 - \frac{\omega_n^2}{\mu_k^2}\right)}{\prod_{k\neq n}^{N-1} \left(1 - \frac{\omega_n^2}{\omega_k^2}\right)} \quad . \tag{3.96'}$$

The reconstruction of the N-beam proceeds as follows:

(i) by means of the interpolation formula (3.59) for $I_o(\omega^2)$ we infer F_o and $\{\lambda_n\}_1^{N-2}$;

(ii) from the quadratic identity we get $K_o(\omega^2)$ and hence $\{\sigma_n\}_1^{N-1}$;

(iii) we compute ℓ_o, f_1 and m_1 by means of (3.74), (3.79) and (3.85);

(iv) we compute λ'_n, μ'_n, ν'_n, σ'_n and ω'_n from (3.92')-(3.96'). This information enables to continue the stripping process.

The above procedure has many advantages over the previous one. It is attractive from the numerical point of view since long divisions as well as locations of zeros are avoided. It is pleasing from the esthetical point of view since it couples the given eigenvalues to the unknown structure without the intermediary of the eigenvalue problem. Finally, it might prove helpful in the search for the missing conditions satisfied by *bona fide* data. Indeed, (3.92)-(3.96) indicate that the relative distribution of eigenvalues determines the slope of the curves $\omega=\lambda_n(x)$, ..., $\omega=\omega_n(x)$ at $x=0$. What should the values of these slopes be in order for the curves not to intersect each other? The answer to this question would provide the missing conditions we are looking for.

The limit $N \to \infty$. We would like to examine whether or not the sequences $\{\rho^{(N)}(x)\}$ and $\{r^{(N)}(x)\}$ converge, where

$$\rho^{(N)}(x) = \sum_{i=1}^{N-1} m_i^{(N)} \delta(x-x^{(N)}) \quad ,$$

and

$$\frac{1}{r^{(N)}(x)} = \sum_{i=1}^{N} f_i^{(N)} \delta(x-x^{(N)}).$$

We shall assume that the beam has a finite length, i.e.

$$L = \lim_{N\to\infty} x_N^{(N)} \quad , \tag{3.97}$$

or, equivalently,

$$L = \lim_{N\to\infty} \sum_{i=0}^{N-1} \ell_i^{(N)} \quad , \tag{3.97'}$$

where

$$\ell_o^{(N)} = \frac{F_2}{F_1} \prod_{n=1}^{N-1} \frac{\nu_n^2}{\mu_n^2} \quad ,$$

$$\ell_1^{(N)} = \frac{F_2'}{F_1'} \prod_{n=1}^{N-2} \frac{\nu'_n{}^2}{\mu'_n{}^2} \quad , \quad \text{etc. . . .}$$

If the data are such that (3.97) holds, then we can define

$$F(x) = \int_x^L \frac{dt}{r(t)} \tag{3.98}$$

and

$$M(x) = \int_x^L (L-t)^2 \rho(t)dt \quad . \tag{3.99}$$

Clearly, the N-th approximation of these functions are

$$F^{(N)}(x) = \sum_{x_i \geq x}^{N} f_i^{(N)} \tag{3.98'}$$

and

$$M^{(N)}(x) = \sum_{x_i \geq x}^{N} (x_N^{(N)} - x_i^{(N)})^2 m_i^{(N)} \quad . \tag{3.99'}$$

From their definitions, $F^{(N)}(x)$ and $M^{(N)}(x)$ are non-increasing functions of x. In fact

$$F^{(N)}(x) \leq F_o^{(N)} \quad . \tag{3.100}$$

We can express $F_o^{(N)}$ in terms of the data by writing (3.59) as follows

$$F_o^{(N)} = \frac{F_1^2}{F_2} + \frac{I_o(0)}{F_2}$$

or better still

$$F_o^{(N)} = \frac{F_1^2}{F_2} + \frac{F_1^2}{F_2} \sum_{n=1}^{N-1} \frac{\prod_{k=1}^{N-1} \left|1 - \frac{\mu_n^2}{\nu_k^2}\right|^2}{\prod_{k=1}^{N-1} \left|1 - \frac{\mu_n^2}{\omega_k^2}\right| \prod_{k \neq n}^{N-1} \left|1 - \frac{\mu_n^2}{\mu_k^2}\right|} . \tag{3.101}$$

Therefore, if

$$\sum_{n=1}^{\infty} \frac{\prod_{k=1}^{\infty} \left|1 - \frac{\mu_n^2}{\nu_k^2}\right|^2}{\prod_{k=1}^{\infty} \left|1 - \frac{\mu_n^2}{\omega_k^2}\right| \prod_{k \neq n}^{\infty} \left|1 - \frac{\mu_n^2}{\mu_k^2}\right|} < \infty , \tag{3.102}$$

Then the limit of $F_o^{(N)}$ as N tends to infinity exists, i.e.

$$F_o = \lim_{N \to \infty} F_o^{(N)} \tag{3.103}$$

and this limit provides an upper bound for $\mathcal{F}^{(N)}(x)$, i.e.

$$\mathcal{F}^{(N)}(x) \leq F_o \quad \text{for} \quad 0 \leq x \leq L \quad \text{and for every N.} \tag{3.104}$$

Similarly, the functions $\{\mathcal{M}^{(N)}(x)\}$ are also bounded above. We can see this as follows. Let us substitute the series representations for $Y(x,\omega^2)$ and $D(x,\omega^2)$, namely

$$Y(x,\omega^2) = -r^2(L)\left[\int_x^L \frac{(t-x)^2}{r}dt + \sum_{n=1}^{\infty} (-1)^n\; y_n(x)\omega^{2n}\right]$$

and

$$D(x,\omega^2) = r^2(L)\left[1 + \sum_{n=1}^{\infty} (-1)^n\; d_n(x)\omega^{2n}\right]$$

in the auxiliary equation (3.25f). Then, the linear terms in ω^2 yield

$$d_1{}' = -\rho \int_x^L \frac{(t-x)^2}{r(t)}\, dt$$

i.e.

$$d_1(x) = \int_x^L \rho(t')dt' \int_x^L \frac{(t-t')^2}{r(t)}\, dt \qquad (3.105)$$

Confronting this result with the product representation of $D(x,\omega^2)$ given in (3.27'e), we conclude that

$$\sum_{n=1}^{\infty} \frac{1}{\omega_n^2} = \int_o^L \rho(t')dt' \int_{t'}^L \frac{(t-t')^2}{r(t)}\, dt$$

or, after interchanging the order of integration

$$\sum_{n=1}^{\infty} \frac{1}{\omega_n^2} = \int_0^L \frac{dt}{r(t)} \int_0^t (t-x)^2 \rho(x)dx. \tag{3.106}$$

$\frac{1}{r(t)}$ being integrable for t in (0,L) we conclude that

$$\int_0^t (t-x)^2 \rho(x)dx < \infty .$$

As a result $M(0)$ exists and

$$M^{(N)}(x) \leq M(0) \text{ for } 0 \leq x \leq L \text{ and for every } N. \tag{3.107}$$

Therefore, the non-increasing sequences of function $\{F^{(N)}(x)\}$ and $\{M^{(N)}(x)\}$ are bounded above. Hence, according to Helly's selection theorem theoren (see e.g. Natanson 1955, p. 220), we can extract from these sequences two subsequences which converge for all x in (0,L) to two non-increasing functions. In view of the uniqueness theorem, these limit functions must be the functions $F(x)$ and $M(x)$ associated with the non-pathological beam. Finally, $\rho(x)$ and $r(x)$ are obtained after differentiation.

CONCLUDING REMARKS

We saw that the information contained in the impulse response is equivalent to the three symphathetic spectra $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$, and the two moments of the flaccidity F_1 and F_2. We also saw that the information contained in the impulse response does not, strictly speaking, determine a single beam. Rather, it determines a class of beams: all the beams in this class have the same oscillating portion, but they differ over that rear portion of their lengths which is stationary. This stationary portion is made up of a weightless rod of infinite rigidity. I was surprised by the fact that the massive wall in which the beams are embedded is not part of the ambiguity.

The uniqueness proof has the drawback of requiring that $\rho(x)$ and $r(x)$ be differentiable functions. This is doubly regrettable: firstly, because the degree of smoothness of $\rho(x)$ and $r(x)$ is not easy to infer from the data and secondly, because discontinuities are likely to occur in the more difficult geophysical problem for which the beam is to serve as a guide. Ideally, these smoothness requirements ought to be relaxed. To that effect, it would be desirable to derive the uniqueness result from the construction, i.e. to show that the approximations form a Cauchy sequence.

Our discussion of the inverse problem was carried out entirely in the frequency domain. Yet, the stripping procedure, which unraveled progressively the structure of the beam, is a reminder of the presence of a wave propagating along the beam. In spite of this time domain intrusion, I believe that the frequency approach is best suited for this inverse problem at hand. Indeed, the continued fractions reliance on the ω-dependence shows very clearly the advantage of working in the frequency domain. In addition, the theories of entire functions and of oscillating kernels provide some very powerful tools for the investigation of the problem in the frequency domain.

The actual construction of the solution required a discretization of the original problem. This discretization is an essential step which cannot be by-passed. Indeed, the global condition (3.102) is a direct result of this apparent detour. The same remark holds for the vibrating string: the global condition (3.3) is a consequence of a discretization followed by the limiting process $N \to \infty$.

The existence of the solution to the inverse problem is tied to very stringent interlacing conditions on $\{\omega_n\}$, $\{\nu_n\}$ and $\{\mu_n\}$. These conditions are such as to guarantee that the spectra $\{\omega_n(x)\}$, ..., $\{\lambda_n(x)\}$ for the truncated beam spanning the interval (x,L), satisfy the grand interlacing. Rather than having to check at each step of the construction whether the grand interlacing is satisfied, it would be desirable to have explicit ways of testing, from the outset, whether the data are *bona fide*.

In the investigation of the limiting process $N \to \infty$, the length L of the beam was assumed to be finite. This convenient assumption is not essential. Very minor modifications are needed to handle the infinite case since the Helly selection theorem can still be used.

Finally, our analysis dealt exclusively with the beam in the free/clamped configuration. For other vibrating configurations, the details of the results would be modified: for instance, the impulse responses would be equivalent to other trio of sympathetic spectra and the gross constants would not necessarily be related to moments of the flaccidity. However, the approach to the questions of uniqueness, existence and construction, which was used in the present paper, ought to be applicable for general boundary conditions.

ACKNOWLEDGMENTS

I would like to thank the Office of Naval Research for supporting this work under Contract N00014-76-C-0034. I would also like to thank Miss Germana Peggion for her help in writing a computer program for implementing the construction procedure.

My understanding of inverse problems has been greatly increased by discussions with various colleagues who have indirectly influenced the present work. At the risk of omitting some, I would like to take this opportunity to thank the following ones: J. K. Cohen, J. D. Cole, J. A. DeSantos, F. Hagin, H. Hochstadt, G. W. Platzman, A. G. Ramm, P. C. Sabatier, M. M. Sondhi, G. Turchetti and C. H. Wilcox.

This work was completed during a visit to the Dept. of Mathematics of the University of Denver. I would like to thank the Chairman of this Department, S. Gudder, for making this visit a very enjoyable one.

Last but not least, I would like to record my indebtedness to Norman Bleistein. The encouragements, insights and criticisms which he provided during the writing process proved invaluable.

FIGURE CAPTIONS

Figure 1: Sketch of the Green's function $N(x,t)$ for the non self-adjoint/clamped configuration.

The curvature of $N(x,t)$ changes at $x = x_o(t)$.

Figure 2: Sketches of the auxiliary function $I(x,\omega^2)$ for a value of ω^2 lying between λ_1^2 and λ_2^2.

The number of local extrema depends on the relation between the λ- and μ- spectra.

Figure 3: The ω^2-x plane. The solid curves $\lambda_1^2(x)$ and $\lambda_2^2(x)$ represent the first and second eigenvalues of the truncated beam spanning the interval (x,L) in the clamped/clamped configuration. The dotted curves represent the corresponding eigenvalues for the supported/clamped configuration. By considering $I(x,\lambda_2^2)$ we can show that this arrangement of eigenvalues is not permissible.

Figure 4: The discrete N-beam. Its structure consists of N clothespins located at x_1, x_2, ..., x_N. These clothespins have masses $\{m_i\}$ and flaccidity $\{f_i\}$ and are connected to each other by weightless, infinitely rigid rods of length $\{\ell_i\}$.

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APPENDICES

<u>Alternative discretization</u>. We still insist that the mass of the beam be concentrated in a few points, namely that

$$\rho(x) = \sum_{i=1}^{N} m_i \, \delta(x-x_i).$$

However, we now allow the connecting segments to have a finite rigidity:

$$\frac{1}{r(x)} = \frac{1}{r_i} \quad \text{for} \quad x_{i-1} \leq x \leq x_i .$$

Substituting these expressions for $\rho(x)$ and $r(x)$ in the auxiliary equations (3.25), we deduce that over the interval (x_{i-1}, x_i)

$$D(x,\omega^2) = D_{i-1}(\omega^2) \; ,$$

$$K(x,\omega^2) = \frac{D_{i-1}(\omega^2)}{r_{i-1}} \cdot (x-x_{i-1}) + K_{i-1}(\omega^2) \; ,$$

$$\Theta(x,\omega^2) = \frac{D_{i-1}(\omega^2)}{2r_{i-1}} \cdot (x-x_{i-1})^2 + K_{i-1}(\omega^2) \cdot (x-x_{i-1})$$

$$+ \Theta_{i-1}(\omega^2) \; ,$$

$$Y(x,\omega^2) = \frac{D_{i-1}(\omega^2)}{3r_{i-1}} \cdot (x-x_{i-1})^3 + K_{i-1}(\omega^2) \cdot (x-x_{i-1})^2$$

$$+2\Theta_{i-1}(\omega^2) \cdot (x-x_{i-1}) + Y_{i-1}(\omega^2) \quad ,$$

$$I(x,\omega^2) = \frac{D_{i-1}(\omega^2)}{12r_{i-1}^{\;2}} \cdot (x-x_{i-1})^4 + \frac{K_{i-1}(\omega^2)}{3r_{i-1}} \cdot (x-x_{i-1})^3$$

$$+ \frac{\Theta_{i-1}(\omega^2)}{r_{i-1}} \quad (x-x_{i-1})^2 + \frac{Y_{i-1}(\omega^2)}{r_{i-1}} \cdot (x-x_{i-1}) + I_{i-1}(\omega^2)$$

We next match the expressions for $I(x,\omega^2)$, ..., $D(x,\omega^2)$ at $x=x_i$:

$$D_i - D_{i-1} = -\omega^2 m_i Y_i$$

$$K_i - K_{i-1} = \frac{\ell_{i-1}}{r_{i-1}} D_{i-1} - \omega^2 m_i I_i$$

$$\Theta_i - \Theta_{i-1} = \ell_{i-1} K_{i-1} + \frac{\ell_{i-1}^{\;2}}{2r_{i-1}} D_{i-1}$$

$$Y_i = Y_{i-1} = 2\ell_{i-1}\Theta_{i-1} + \ell_{i-1}^{\;2} K_{i-1} + \frac{\ell_{i-1}^{\;3}}{3r_{i-1}} D_{i-1}$$

$$I_i - I_{i-1} = \frac{\ell_{i-1}}{r_{i-1}} Y_{i-1} + \frac{\ell_{i-1}^{\;2}}{r_{i-1}} \Theta_{i-1} + \frac{\ell_{i-1}^{\;3}}{3r_{i-1}} K_{i-1} + \frac{\ell_{i-1}^{\;4}}{12r_{i-1}^2} D_{i-1} \quad .$$

These are the discrete auxiliary equations. It should be noted that they satisfy the following identity:

$$-I_iD_i + Y_iK_i - \Theta_i^2 = -I_{i-1}D_{i-1} + Y_{i-1}K_{i-1} - \Theta_{i-1}^2 \quad .$$

This identity leads to the quadratic identity when coupled with the boundary conditions.

Example of a violation of the grand interlacing. Let us consider a 4-beam whose impulse response satisfies the small interlacing. More specifically, let

$$\omega_1^2 = 1.00, \qquad \nu_1^2 = 2.00, \qquad \mu_1^2 = 3.00 \ ,$$

$$\omega_2^2 = 3.01, \qquad \nu_2^2 = 4.00, \qquad \mu_2^2 = 5.00 \ ,$$

$$\omega_3^2 = 6.00, \qquad \nu_3^2 = 7.00, \qquad \mu_3^2 = 8.00 \ ,$$

and

$$F_1 = 1.00$$

$$F_2 = 1.33 \ .$$

Following the method outlined in §3, we can construct $I_o(\omega^2)$ which is a polynomial of degree 2 in ω^2. The zeros of $I_o(\omega^2)$ turn out to be

$$\lambda_1^2 = 4.94$$

and

$$\lambda_2^2 = 7.90$$

Note that

$$\lambda_1^2 > \nu_2^2$$

and

$$\lambda_2^2 > \nu_3^2$$

which violates the grand interlacing. Were we to pursue the construction of this 4-beam, we would find that some of the ℓ's, m's and f's are negative.

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